Fluid Mechanics - MTF053

Lecture 11

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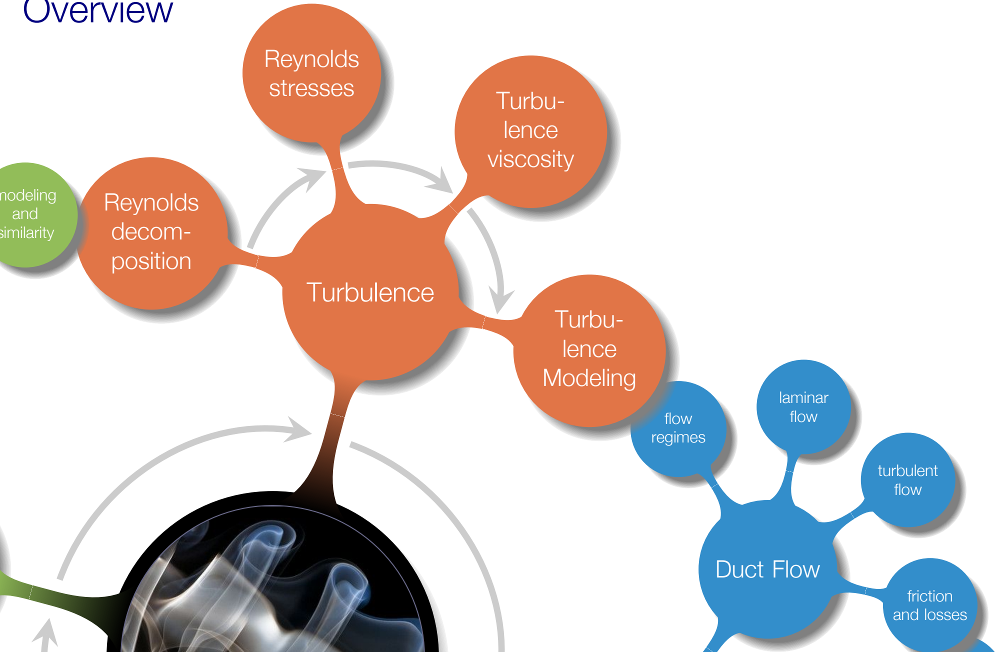
Chapter 6 - Viscous Flow in Ducts

Overview


Reynolds stresses
Turbu-lence viscosity
modeling and similarity
Reynolds decom-position
Turbulence
Turbu-lence Modeling
flow regimes
laminar flow
turbulent flow
Duct Flow
friction and losses

Learning Outcomes

3 **Define** the Reynolds number

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

6 **Explain** what a boundary layer is and when/where/why it appears

8 **Understand** and be able to **explain** the concept shear stress

18 **Explain** losses appearing in pipe flows

19 **Explain** the difference between laminar and turbulent pipe flow

20 **Solve** pipe flow problems using Moody charts

24 **Explain** what is characteristic for a turbulent flow

25 **Explain** Reynolds decomposition and derive the RANS equations

26 **Understand** and **explain** the Boussinesq assumption and turbulent viscosity

27 **Explain** the difference between the regions in a boundary layer and what is characteristic for each of the regions (viscous sub layer, buffer region, log region)

if you think about it, pipe flows are everywhere (a pipe flow is not a flow of pipes)

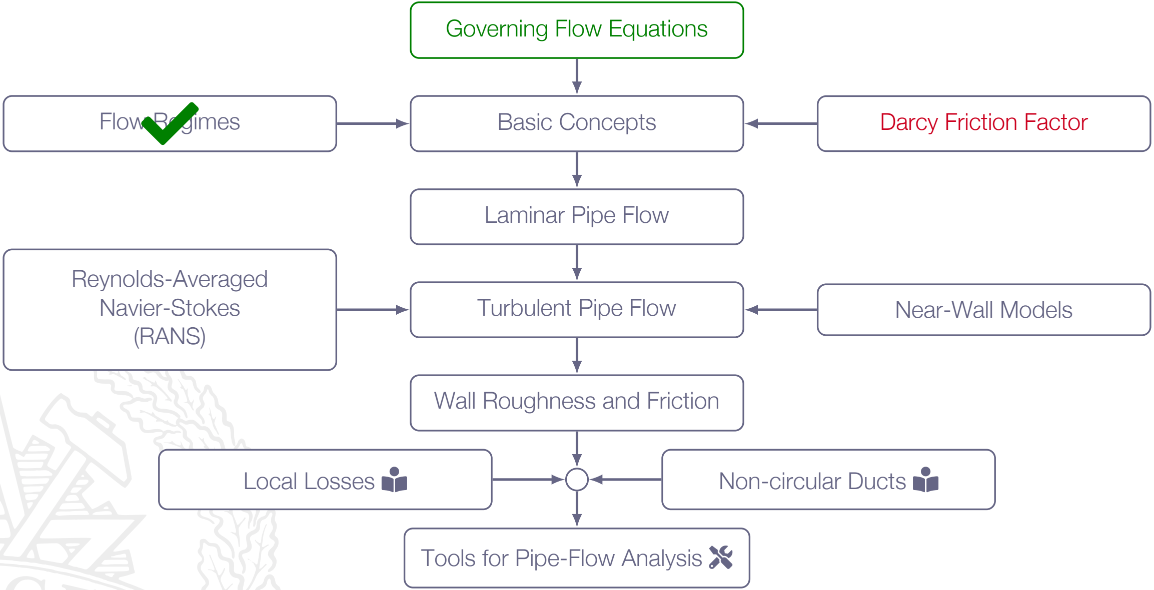
Complementary Course Material

These lecture notes covers chapter 6 in the course book and additional course material that you can find in the following documents

`MTF053_Equation-for-Boundary-Layer-Flows.pdf`

`MTF053_Turbulence.pdf`

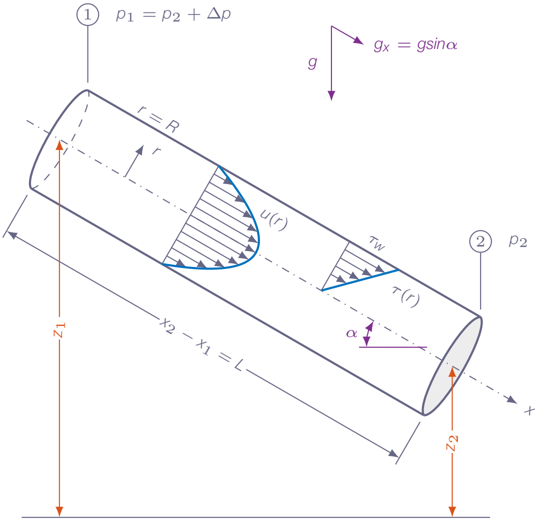
Roadmap - Viscous Flow in Ducts



Head Loss

Assumptions:

1. steady-state flow
2. incompressible
3. fully developed
4. no pumps or turbines



Head Loss

Continuity gives:

$$Q_1 = Q_2 = Q,\ V_1 = V_2 = V_{av}$$

Energy equation for steady flow without pumps or turbines:

$$\left(\frac{p}{\rho g} + \alpha\frac{V^2}{2g} + z\right)_1 = \left(\frac{p}{\rho g} + \alpha\frac{V^2}{2g} + z\right)_2 + h_f$$

Fully developed flow $\Rightarrow \alpha_1 = \alpha_2$

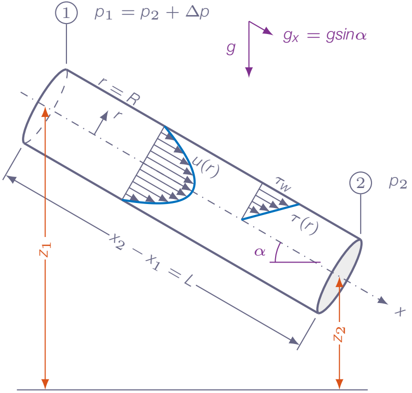
$$h_f = (z_1 - z_2) + \left(\frac{p_1 - p_2}{\rho g}\right) = \Delta z + \frac{\Delta p}{\rho g}$$

Head Loss

Apply the momentum equation along the pipe:

$$\sum F_x = \Delta p(\pi R^2) + \rho g(\pi R^2)L\sin\alpha - \tau_w(2\pi R)L$$

$$\sum F_x = \dot{m}(V_2 - V_1) = 0$$



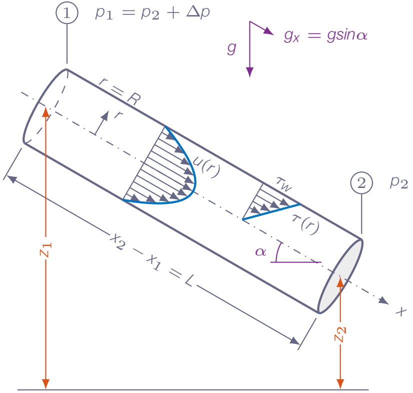
Head Loss

$$\Delta p(\pi R^2) + \rho g(\pi R^2) L \sin\alpha = \tau_w(2\pi R)L$$

$$\frac{\Delta p}{\rho g} + L\sin\alpha = \frac{2\tau_w}{\rho g}\frac{L}{R}$$

$$\frac{\Delta p}{\rho g} + \Delta z = \frac{4\tau_w}{\rho g}\frac{L}{d}$$

$$h_f = \frac{4\tau_w}{\rho g}\frac{L}{d}$$



Friction Factor

Henry Darcy 1803-1858

$$h_f = f_D \frac{L}{d} \frac{V_{av}^2}{2g}$$

where

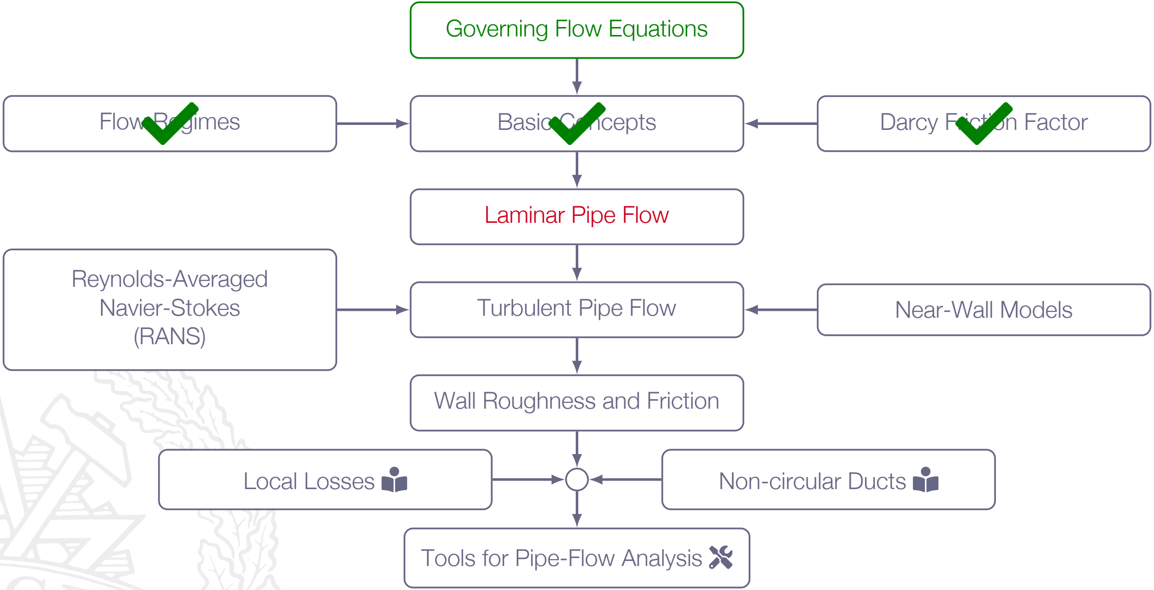
$$f_D = f(Re_d, \varepsilon/d, duct\ shape)$$

is the **Darcy friction factor**

$$\frac{4\tau_w}{\rho g}\frac{L}{d} = f_D \frac{L}{d}\frac{V_{av}^2}{2g} \Rightarrow f_D = \frac{8\tau_w}{\rho V_{av}^2}$$

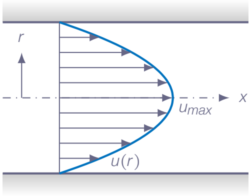
Note! for non-circular pipes, τ_w is an average value around the duct perimeter

Roadmap - Viscous Flow in Ducts



Fully-Developed Laminar Pipe Flow

Pressure driven (Poiseuille flow) in a circular pipe with the diameter D and radius R



Assumptions:

1. Steady state
2. Incompressible
3. Laminar
4. Fully developed

$$u(r) = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \Rightarrow \frac{du}{dr} = -2u_{max}\frac{r}{R^2} = \left\{V_{av} = \frac{u_{max}}{2}\right\} = -4V_{av}\frac{r}{R^2}$$

$$\tau_w = \mu\left|\frac{du}{dr}\right|_{r=R} = \frac{4\mu V_{av}}{R} = \frac{8\mu V_{av}}{D}$$

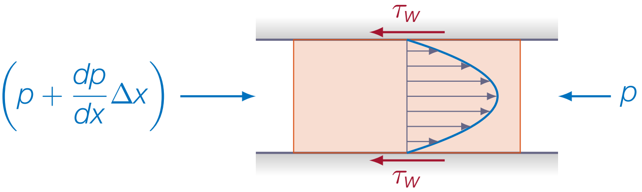
Fully-Developed Laminar Pipe Flow

For laminar flow:

$$f_D = \frac{8\tau_w}{\rho V_{av}^2} = \left\{\tau_w = \frac{8\mu V_{av}}{D}\right\} = \frac{64\mu}{\rho V_{av} D} = \frac{64}{Re_D}$$

Note! in laminar flow, the friction factor is inversely proportional to the Reynolds number

Fully-Developed Laminar Pipe Flow



$$\left(p + \frac{dp}{dx}\Delta x\right)\pi R^2 - p\pi R^2 - \tau_w 2\pi R\Delta x = 0 \Rightarrow$$

$$\frac{dp}{dx} = 2\frac{\tau_w}{R} = \left\{\tau_w = \mu \left.\frac{du}{dr}\right|_{r=R}\right\} = \frac{2\mu}{R}\left.\frac{du}{dr}\right|_{r=R}$$

$$u(r) = u_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) \Rightarrow \frac{du}{dr} = 2u_{max}\frac{r}{R^2}$$

$$\frac{dp}{dx} = \frac{4\mu}{R^2}u_{max} \Leftrightarrow u_{max} = -\frac{R^2}{4\mu}\frac{dp}{dx}$$

Fully-Developed Laminar Pipe Flow

$$-\frac{dp}{dx} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right) \Rightarrow u_{max} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{4\mu}$$

$$V_{av} = \frac{u_{max}}{2} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{R^2}{8\mu}$$

$$Q = \int u dA = V_{av} A = V_{av}\frac{\pi D^2}{4} = \left(\frac{\Delta p + \rho g \Delta z}{L}\right)\frac{\pi D^4}{128\mu}$$

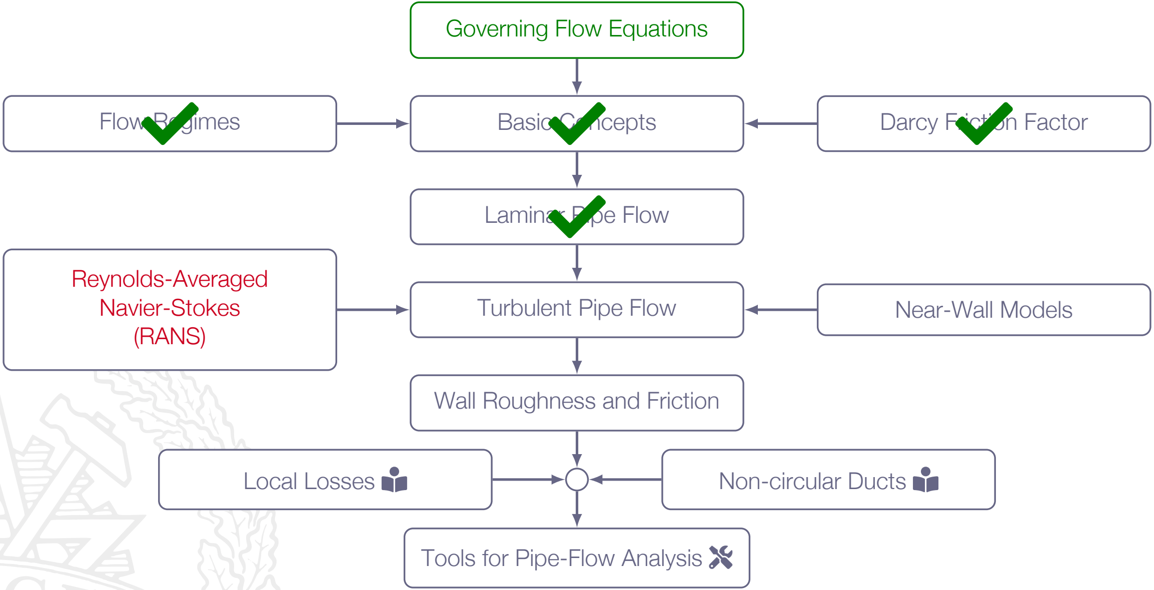
Fully-Developed Laminar Pipe Flow

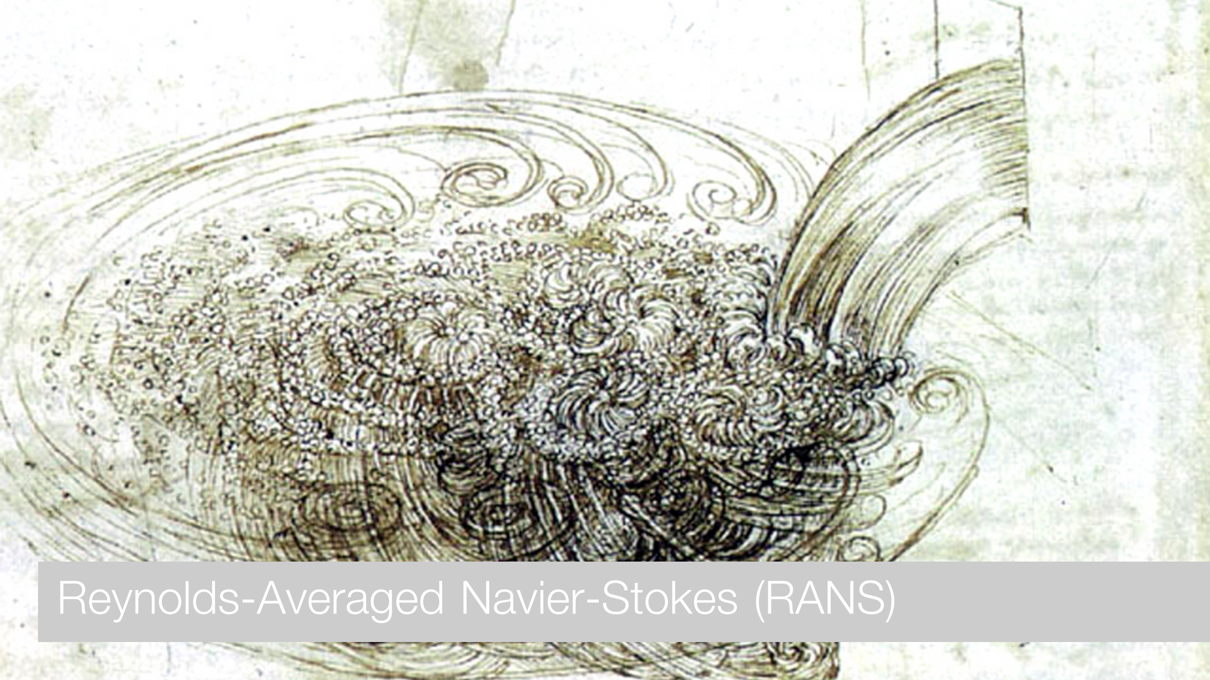
We can now calculate the head loss according to

$$h_f = f_D \frac{L}{D} \frac{V_{av}^2}{2g} \text{ where } f_D = \frac{8\tau_w}{\rho V_{av}^2}$$

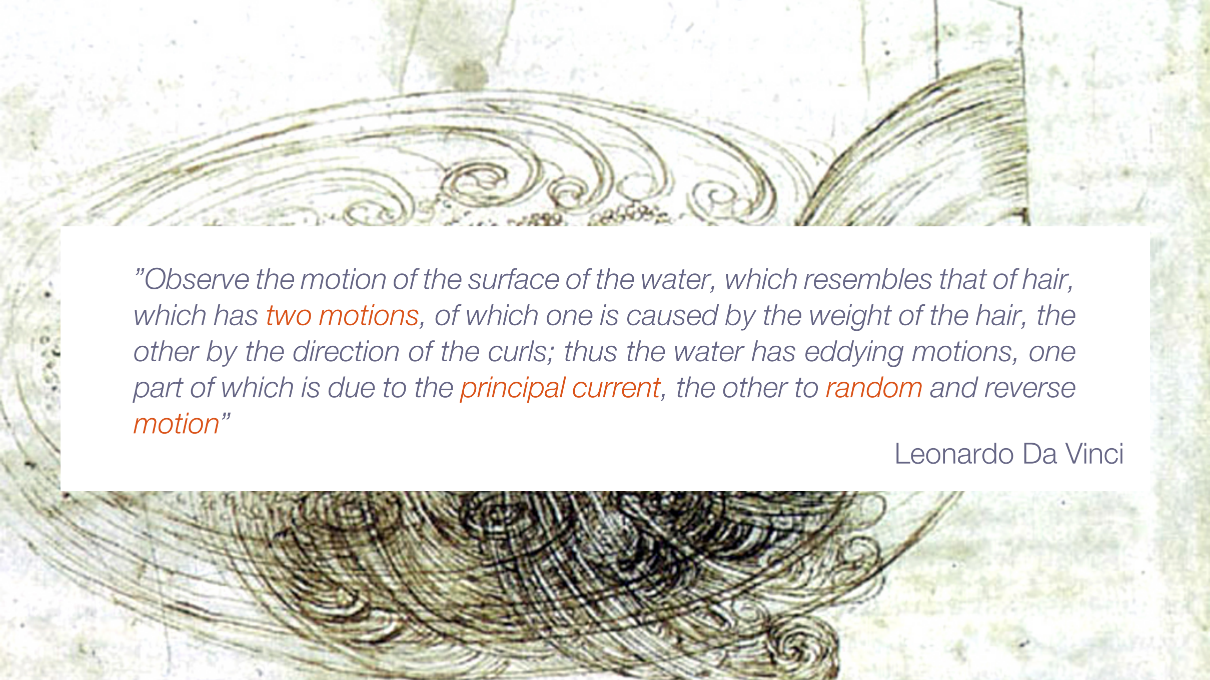
$$h_f = \frac{4\tau_w L}{\rho g D} = \left\{ \tau_w = \frac{8\mu V_{av}}{D} \right\} = \frac{16\mu V_{av} L}{\rho g D R} = \frac{32\mu V_{av} L}{\rho g D^2} = \left\{ V_{av} = \frac{4Q}{\pi D^2} \right\} = \frac{128\mu Q L}{\pi \rho g D^4}$$

Roadmap - Viscous Flow in Ducts





Reynolds-Averaged Navier-Stokes (RANS)



"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion"

Leonardo Da Vinci

Governing Equations

Assumptions:

1. constant density and viscosity
2. no thermal interaction

Flow equations:

continuity: $\nabla \cdot \mathbf{V} = 0$

momentum: $\rho \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V}$

Governing Equations

The differential energy equation is not included here but let's have a look at it anyway

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

Pressure work:

pressure drives the flow through the duct

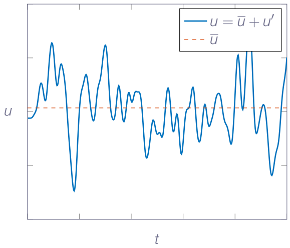
Viscous work:

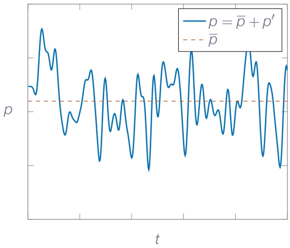
no-slip condition $\Rightarrow$ zero velocity at the walls $\Rightarrow$ no work done by wall shear stress

So, where does the energy go?

pressure work is balanced by **viscous dissipation** in the interior of the flow

Reynolds’ Decomposition

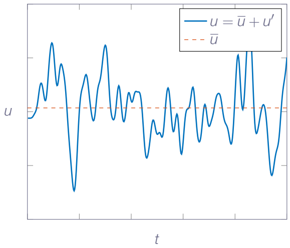


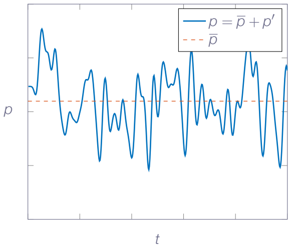


Not possible to solve analytically

Often, the time-averaged quantities are what we are looking for

Reynolds' Decomposition





$$\overline{u} = \frac{1}{T}\int_0^T u dt$$

$$u' = u - \overline{u}$$

$$\overline{u'} = \frac{1}{T}\int_0^T (u - \overline{u}) dt = \overline{u} - \overline{u} = 0$$

Reynolds' Decomposition

The mean square of the fluctuations are, however, not zero

$$\overline{u'^2} = \frac{1}{T}\int_0^T u'^2 dt \neq 0$$

measure of turbulence intensity

Mean of fluctuation products are generally not zero ($\overline{u'v'}$, $\overline{u'p'}$)

Reynolds' Decomposition

Reynolds' idea was to split all properties into mean and fluctuating parts:

$$u = \overline{u} + u', \; v = \overline{v} + v', \; w = \overline{w} + w', \; p = \overline{p} + p'$$

1. insert into the governing equations
2. time average the equations

Reynolds-Averaged Navier Stokes (RANS)

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum (x-component):

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

Reynolds-Averaged Navier Stokes (RANS)

Continuity:

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} + \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

time averaging the equation gives

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} + \frac{\partial \overline{w}}{\partial z} = 0$$

and as a consequence

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

Reynolds-Averaged Navier Stokes (RANS)

Momentum (x-component):

$$
\begin{aligned}
&\rho\left(\frac{\partial\overline{u}}{\partial t}+\frac{\partial u'}{\partial t}\right)+\\
&\rho\left(\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{u}\frac{\partial u'}{\partial x}+u'\frac{\partial\overline{u}}{\partial x}+u'\frac{\partial u'}{\partial x}\right)+\\
&\rho\left(\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{v}\frac{\partial u'}{\partial y}+v'\frac{\partial\overline{u}}{\partial y}+v'\frac{\partial u'}{\partial y}\right)+\\
&\rho\left(\overline{w}\frac{\partial\overline{u}}{\partial z}+\overline{w}\frac{\partial u'}{\partial z}+w'\frac{\partial\overline{u}}{\partial z}+w'\frac{\partial u'}{\partial z}\right)=\\
&-\frac{\partial\overline{p}}{\partial x}-\frac{\partial p'}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2\overline{u}}{\partial x^2}+\frac{\partial^2\overline{u}}{\partial y^2}+\frac{\partial^2\overline{u}}{\partial z^2}+\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}+\frac{\partial^2 u'}{\partial z^2}\right)
\end{aligned}
$$

Reynolds-Averaged Navier Stokes (RANS)

Momentum (x-component):

time averaging the equation gives:

$$\rho \left(\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} + \overline{w}\frac{\partial \overline{u}}{\partial z} + \overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} \right) =$$
$$-\frac{\partial \overline{p}}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} + \frac{\partial^2 \overline{u}}{\partial z^2} \right)$$

The highlighted terms can be rewritten as:

$$\overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} + \overline{w'\frac{\partial u'}{\partial z}} = \frac{\partial \overline{u'u'}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z} - \overline{u' \underbrace{\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right)}_{=0}}$$

Reynolds-Averaged Navier Stokes (RANS)

the continuity equation reduces to

$$\boxed{\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0}$$

the axial component of the momentum equation:

$$\boxed{\begin{aligned}\rho\frac{D\bar{u}}{Dt}=-\frac{\partial \bar{p}}{\partial x}+\rho g_x+\frac{\partial}{\partial x}\left(\mu\frac{\partial \bar{u}}{\partial x}-\rho\overline{u'^2}\right)+\\ \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial \bar{u}}{\partial z}-\rho\overline{u'w'}\right)\end{aligned}}$$

Reynolds-Averaged Navier Stokes (RANS)

By applying Reynolds' decomposition to our governing equations, we have introduced a number of new unknowns

The number of equations is the same as before, which means problems

Our new problem has a name

The closure problem

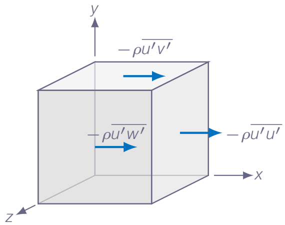
Reynolds Stresses

The three correlation terms $-\rho\overline{u'^2}$, $-\rho\overline{u'v'}$, and $-\rho\overline{u'w'}$ are called Reynolds stresses or turbulent stresses

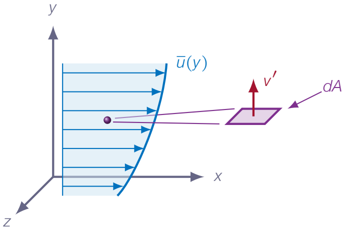
In duct and boundary layer flow, the stress $-\rho\overline{u'v'}$, associated with the direction normal to the wall, is dominant

$$\rho\frac{D\bar{u}}{Dt} \approx -\frac{\partial\bar{p}}{\partial x} + \rho g_x + \frac{\partial\tau}{\partial y}$$

$$\tau = \mu\frac{\partial\bar{u}}{\partial y} - \rho\overline{u'v'} = \tau_{lam} + \tau_{turb}$$



Reynolds Stresses



mass flow through surface element: $\dot{m}_y = \rho v' dA$

momentum balance in x-direction: $F_x = \dot{m}_y u = \rho v' \left(\overline{u} + u'\right) dA$

$$\tau_{dA} = -\frac{\overline{F}_x}{dA} = -\rho\overline{v'\left(\overline{u} + u'\right)} = -\rho\overline{v'\overline{u}} - \rho\overline{u'v'} = \left\{\overline{v'\overline{u}} = \overline{v'}\overline{u} = 0\right\} = -\rho\overline{u'v'}$$

$\Rightarrow -\rho\overline{u'v'}$ can be interpreted as a shear stress

Reynolds Stresses

Introducing **turbulent viscosity** μ_t defined such that

$$-\rho\overline{u'v'} = \mu_t \frac{\partial \overline{u}}{\partial y}$$

Boussinesq's assumption

With the turbulent viscosity, the total shear stress τ becomes:

$$\tau = \mu \frac{\partial \overline{u}}{\partial y} - \rho\overline{u'v'} = (\mu + \mu_t)\frac{\partial \overline{u}}{\partial y}$$

Laminar vs Turbulent Shear Stress

laminar shear (τ_{lam}) dominates in the near-wall region

turbulent shear (τ_{turb}) dominates in the outer region

both are important in the overlap layer