Fluid Mechanics - MTF053

Lecture 7

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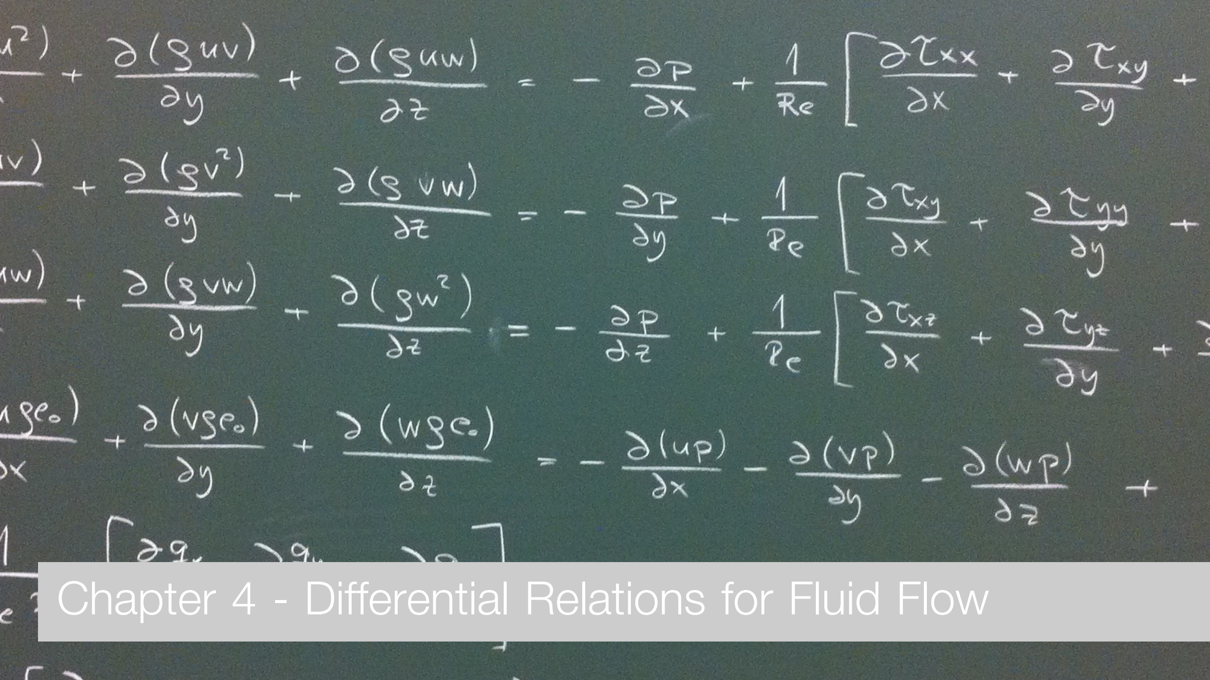


Chapter 4 - Differential Relations for Fluid Flow

Overview

rotation
stream function
conser-vation relations
Differential Relations
The Pi theorem
non-dimensional equations
modeling and similarity
Dimen-sional Analysis
Flow Relations
Bernoulli
Integral Relations
conser-vation relations
Reynolds
buoyancy
hydrostatic forces

Learning Outcomes

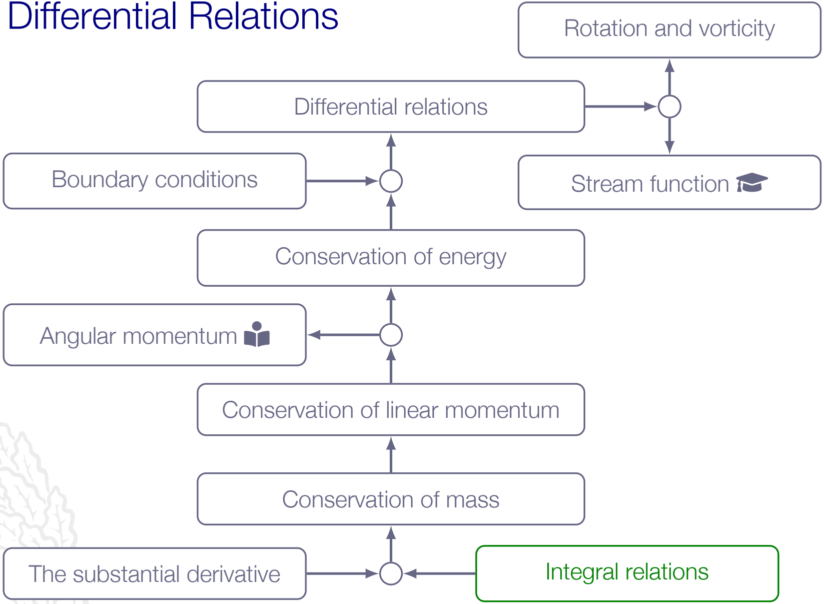
4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

14 **Derive** the continuity, momentum and energy equations on differential form

36 **Define** and explain vorticity

let's push the control volume approach to the limit ...

Roadmap - Differential Relations



Differential Relations

seeking the point-by-point details of a flow pattern by analyzing an infinitesimal region of the flow

Differential Relations

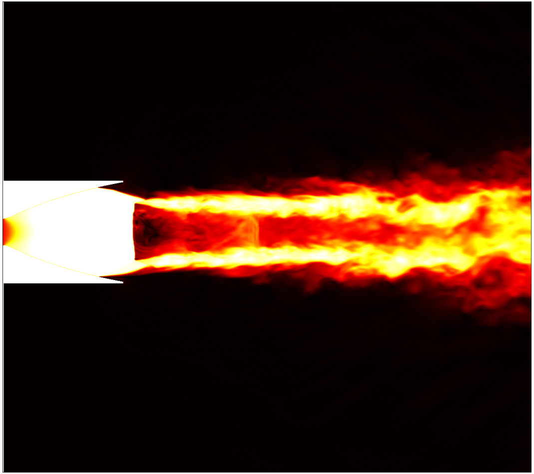
Apply the four basic conservation laws to an infinitesimal control volume

The differential relations are in general very difficult to solve

Analytical solutions exists for a few cases

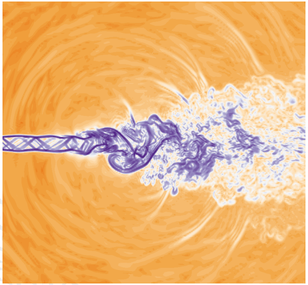
The differential relations form the basis for CFD software

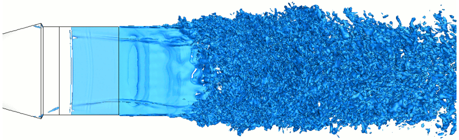
High-Speed Nozzle Flow



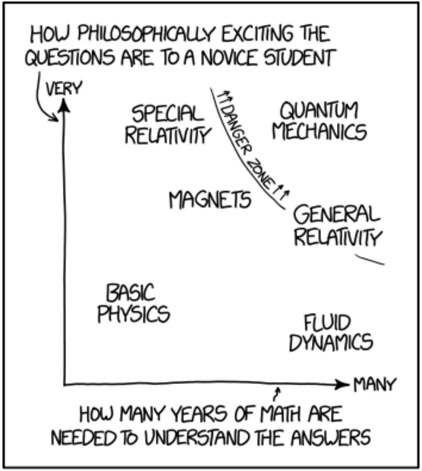
The Acoustic Signature of a Supersonic Jet

Screeching rectangular supersonic jet

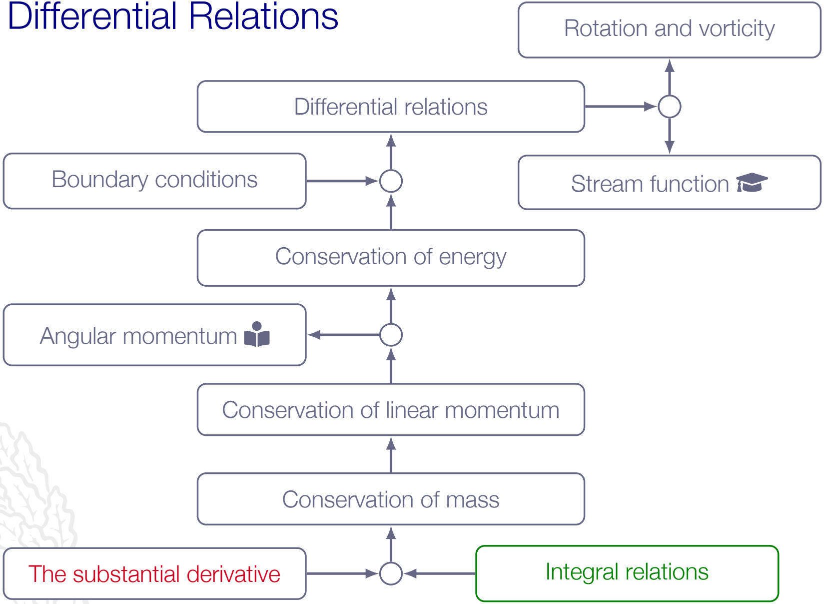




Lots of Equations Ahead



Roadmap - Differential Relations



Frame of Reference

Eulerian: observer fixed in space

Lagrangian: observer follows a fluid particle

recall the speedometer/traffic-camera analogy

Acceleration Field

In order to get to Newton's second law, we need the acceleration vector

Velocity field:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{e}_x u(x, y, z, t) + \mathbf{e}_y v(x, y, z, t) + \mathbf{e}_z w(x, y, z, t)$$

Acceleration field:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{e}_x \frac{du}{dt} + \mathbf{e}_y \frac{dv}{dt} + \mathbf{e}_z \frac{dw}{dt}$$

Acceleration Field

Each scalar component of the velocity vector (u, v, w) is a function of four variables (x, y, z, t) and thus

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

By definition $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

Acceleration Field

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V}\cdot\nabla)v$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w$$

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{D\mathbf{V}}{Dt}$$

The Substantial derivative

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{local} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{convective} = \underbrace{\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}}_{substantial\ derivative} = \frac{D\mathbf{V}}{Dt}$$

local acceleration $(\partial \mathbf{V}/\partial t)$:
only non-zero in unsteady flows (always zero in steady-state flows)

convective acceleration $((\mathbf{V}\cdot\nabla)\mathbf{V})$:
non-zero for fluid particles that moves through regions of spatially varying velocity

the substantial derivative $(D\mathbf{V}/Dt)$:
operator that combines local and convective acceleration

The Substantial derivative

$$\frac{D\varphi}{Dt}$$

the sum of the **local** derivative and the **convective** derivative

follows a fluid particle but is expressed in an **Eulerian frame of reference**

an **operator** that can be applied to any variable φ

example: the substantial deriviative applied to pressure

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p$$

Roadmap - Differential Relations

Rotation and vorticity

Differential relations

Boundary conditions

Stream function

Conservation of energy

Angular momentum

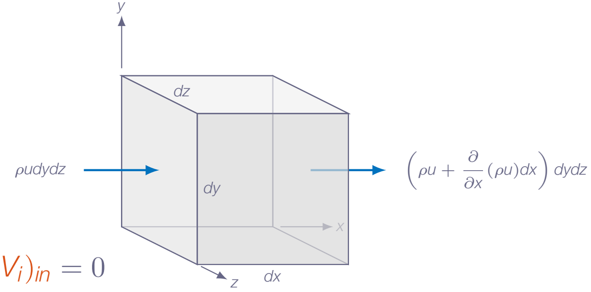
Conservation of linear momentum

Conservation of mass

The substantial derivative

Integral relations

Mass Conservation



Integral form:

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

Infinitesimal control volume:

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dxdydz$$

$$\sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} \approx \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz = 0$$

Mass Conservation

The result is the **continuity equation** - conservation of mass for an infinitesimal control volume

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

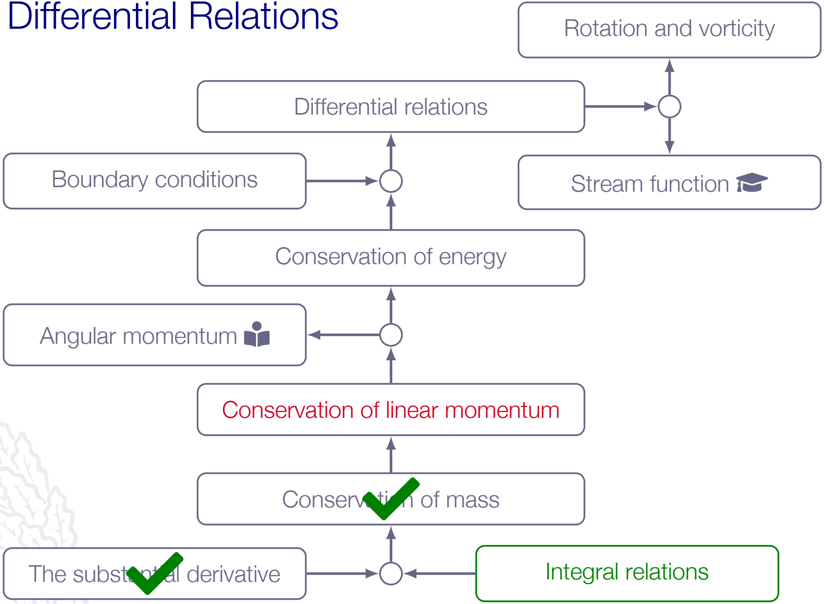
or in more compact form using vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

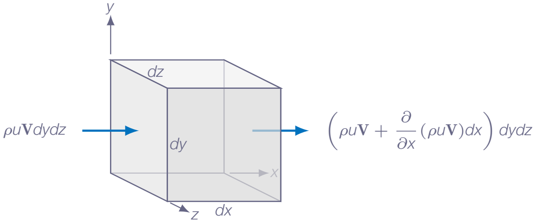
Incompressible flow (constant density)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} = 0$$

Roadmap - Differential Relations



Linear Momentum



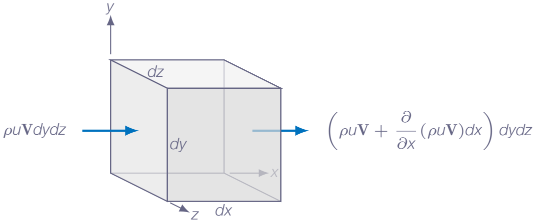
Integral form:

$$\sum \mathbf{F} = \int_{CV} \frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} + \sum(\dot{m}_i \mathbf{V}_i)_{out} - \sum(\dot{m}_i \mathbf{V}_i)_{in}$$

Infinitesimal control volume:

$$\frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} \approx \frac{\partial}{\partial t}(\mathbf{V}\rho)dxdydz$$

Linear Momentum



Face	Inlet momentum flux	Outlet momentum flux
constant x	$\rho u \mathbf{V} dydz$	$\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V}) dx\right] dydz$
constant y	$\rho v \mathbf{V} dxdz$	$\left[\rho v \mathbf{V} + \frac{\partial}{\partial y}(\rho v \mathbf{V}) dy\right] dxdz$
constant z	$\rho w \mathbf{V} dxdy$	$\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V}) dz\right] dxdy$

Linear Momentum

Integral form:

$$\sum \mathbf{F} = \int_{CV} \frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} + \sum(\dot{m}_i\mathbf{V}_i)_{out} - \sum(\dot{m}_i\mathbf{V}_i)_{in}$$

Infinitesimal control volume:

$$\sum \mathbf{F} = \frac{\partial}{\partial t}(\mathbf{V}\rho)dxdydz + \left[\frac{\partial}{\partial x}(\rho u\mathbf{V}) + \frac{\partial}{\partial y}(\rho v\mathbf{V}) + \frac{\partial}{\partial z}(\rho w\mathbf{V})\right] dxdydz$$

Linear Momentum

$$\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\rho)}_{\mathbf{V}\frac{\partial \rho}{\partial t}+\rho\frac{\partial \mathbf{V}}{\partial t}} + \underbrace{\frac{\partial}{\partial x}(\rho u\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial x}(\rho u)+\rho u\frac{\partial \mathbf{V}}{\partial x}} + \underbrace{\frac{\partial}{\partial y}(\rho v\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial y}(\rho v)+\rho v\frac{\partial \mathbf{V}}{\partial y}} + \underbrace{\frac{\partial}{\partial z}(\rho w\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial z}(\rho w)+\rho w\frac{\partial \mathbf{V}}{\partial z}}$$

can be rewritten as

$$\mathbf{V}\underbrace{\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V})\right]}_{\text{continuity equation}} + \rho\underbrace{\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=\frac{D\mathbf{V}}{Dt}}$$

and thus

$$\sum \mathbf{F} = \rho\frac{D\mathbf{V}}{Dt}dxdydz$$

Linear Momentum - Forces

$$\sum \mathbf{F} = \rho \frac{D\mathbf{V}}{Dt} dxdydz$$

The net force equals mass times acceleration where the acceleration is obtained using the **substantial derivative** - the sum of local and convective acceleration

$\sum \mathbf{F}:$

body forces: gravity and other field forces

surface forces: pressure and viscous stresses

Linear Momentum - Gravity Force

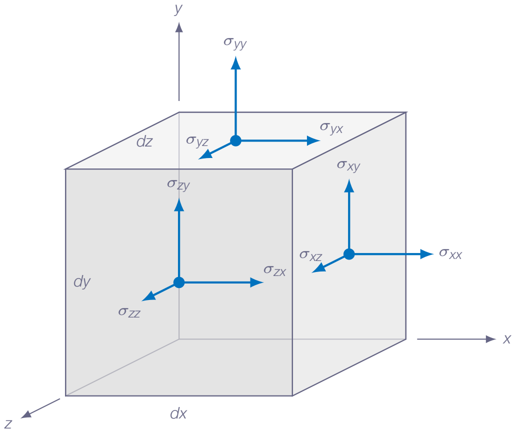
$$d\mathbf{F}_{gravity} = \rho\mathbf{g}dxdydz$$

if gravity is aligned with the negative z-direction

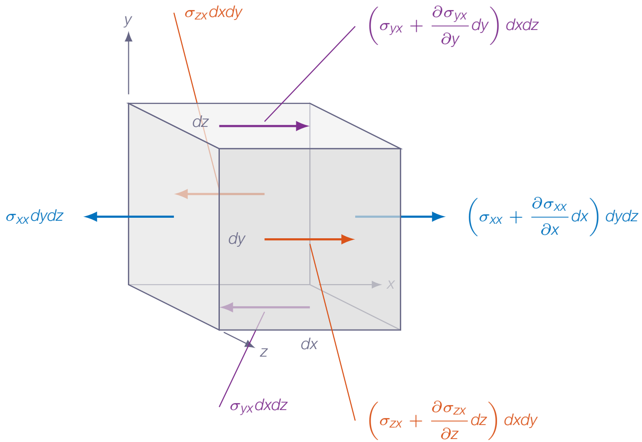
$$d\mathbf{F}_{gravity} = -\mathbf{e}_z\rho g dxdydz$$

Linear Momentum - Surface Forces

$$\sigma_{ij} = \begin{vmatrix} -p+\tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz} \end{vmatrix}$$



Linear Momentum - Surface Forces



$$dF_{x,surf}=\left[\frac{\partial}{\partial x}(\sigma_{xx})+\frac{\partial}{\partial y}(\sigma_{yx})+\frac{\partial}{\partial z}(\sigma_{zx})\right]dxdydz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz$$

$$\sigma_{xx} = \tau_{xx} - p, \ \sigma_{yx} = \tau_{yx}, \ \sigma_{zx} = \tau_{zx}$$

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right] dxdydz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \right] dxdydz$$

$$dF_{y,surf} = \left[-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy}) \right] dxdydz$$

$$dF_{z,surf} = \left[-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz}) \right] dxdydz$$

Linear Momentum - Surface Forces

$$d\mathbf{F} = \left[-\nabla p + \nabla \cdot \tau_{ij}\right] dxdydz$$

where

$$\tau_{ij} = \begin{vmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{vmatrix}$$

is the viscous stress tensor

Linear Momentum - Forces

Now, inserting the forces into the momentum equation gives

$$\boxed{\rho\mathbf{g} - \nabla p + \nabla \cdot \tau_{ij} = \rho\frac{D\mathbf{V}}{Dt}}$$

Linear Momentum

vector notation is powerful, tensor notation is even better ...

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Note! the convective term (RHS) is nonlinear

Linear Momentum - Viscous Stresses in a Newtonian Fluid

Recall:

"For a Newtonian fluid, the viscous stresses are proportional to the element strain and the viscosity"

For incompressible flow:

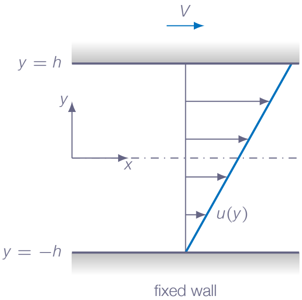
$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x},\ \tau_{yy} = 2\mu\frac{\partial v}{\partial y},\ \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Linear Momentum - Shear Stress Components in a Couette Flow

Couette flow: $u = u(y)$



$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} = 0$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} = 0$$

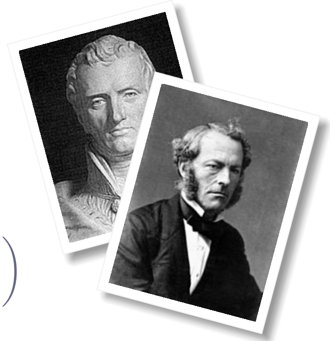
$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} = 0$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{\partial u}{\partial y}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

Linear Momentum



The incompressible Navier-Stokes equations

$$
\begin{aligned}
\rho\frac{Du}{Dt} &= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \\
\rho\frac{Dv}{Dt} &= \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\
\rho\frac{Dw}{Dt} &= \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}
$$

Linear Momentum

The incompressible Navier-Stokes equations

Non-linear equations

Three equations and four unknowns (p, u, v, w)

Combined with the continuity equations we have four equations and four unknowns

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS EVENTS EUCLID

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

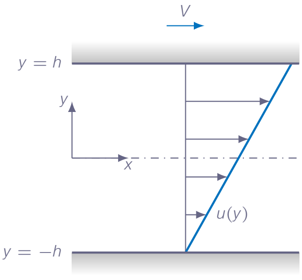
Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Example - Couette Flow



Assumptions:

1. incompressible ($\rho = const$)
2. steady-state
3. lower plate fixed, upper plate moving with the velocity V
4. flow only in the x-direction $v = w = 0,\ u \neq 0$
5. no pressure gradient

Example - Couette Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0,\ \frac{\partial p}{\partial x} = 0\} \Rightarrow \mu\frac{\partial^2 u}{\partial y^2} = 0$$

Example - Couette Flow

$$\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u = ay + b$$

boundary conditions:

$$\left.\begin{array}{l} u(h) = V \\ u(-h) = 0 \end{array}\right\} \Rightarrow$$

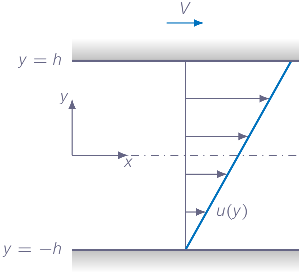
$$\begin{array}{rrcrcr} & V & = & ah & + & b \\ + & 0 & = & -ah & + & b \\ \hline & V & = & & & 2b \end{array}$$

$$b = \frac{V}{2}$$

$$\begin{array}{rrcrcr} & V & = & ah & + & b \\ - & 0 & = & -ah & + & b \\ \hline & V & = & 2ah & & \end{array}$$

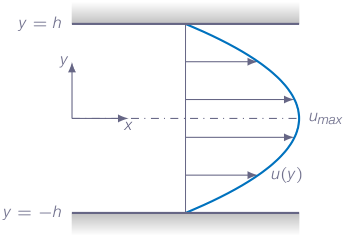
$$a = \frac{V}{2h}$$

Example - Couette Flow



$$u = \frac{V}{2}\left(\frac{y}{h} - 1\right)$$

Example - Poiseuille Flow



Assumptions:

1. incompressible ($\rho = const$)
2. steady-state
3. lower and upper plate fixed
4. flow only in the x-direction $v = w = 0, \; u \neq 0$
5. pressure gradient driven

Example - Poiseuille Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0\} \Rightarrow \mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

Example - Poiseuille Flow

momentum equation (y-direction and z-direction):

$$\left.\begin{aligned} \frac{\partial p}{\partial y} &= 0 \\[2ex] \frac{\partial p}{\partial z} &= 0 \end{aligned}\right\} \Rightarrow p = p(x)$$

Example - Poiseuille Flow

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = const < 0$$

Why constant?

1. RHS function of x only
2. LHS function of y only
3. RHS=LHS $\Rightarrow$ must be a constant

Why < 0?

pressure must decrease in the flow direction

Example - Poiseuille Flow

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = const < 0$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + ay + b$$

Example - Poiseuille Flow

boundary conditions:

$$\left.\begin{array}{l} u(h) = 0 \\ u(-h) = 0 \end{array}\right\} \Rightarrow$$

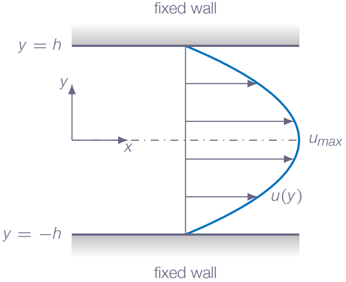
$$\begin{array}{rrcccccc} & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & + & ah & + & b \\ + & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & - & ah & + & b \\ \hline & 0 & = & \frac{1}{\mu}\frac{dp}{dx}h^2 & & & + & 2b \end{array}$$

$$b = -\frac{1}{2\mu}\frac{dp}{dx}h^2$$

$$\begin{array}{rrcccccc} & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & + & ah & + & b \\ - & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & - & ah & + & b \\ \hline & 0 & = & & & 2ah & & \end{array}$$

$$a = 0$$

Example - Poiseuille Flow



$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \left(\frac{y}{h}\right)^2\right)$$

$$\frac{du}{dy} = \frac{dp}{dx}\frac{y}{\mu} \Rightarrow \left.\frac{du}{dy}\right|_{y=0} = 0 \Rightarrow u_{max} = u(0) = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

(remember: $dp/dx < 0$)