Fluid Mechanics - MTF053

Chapter 4

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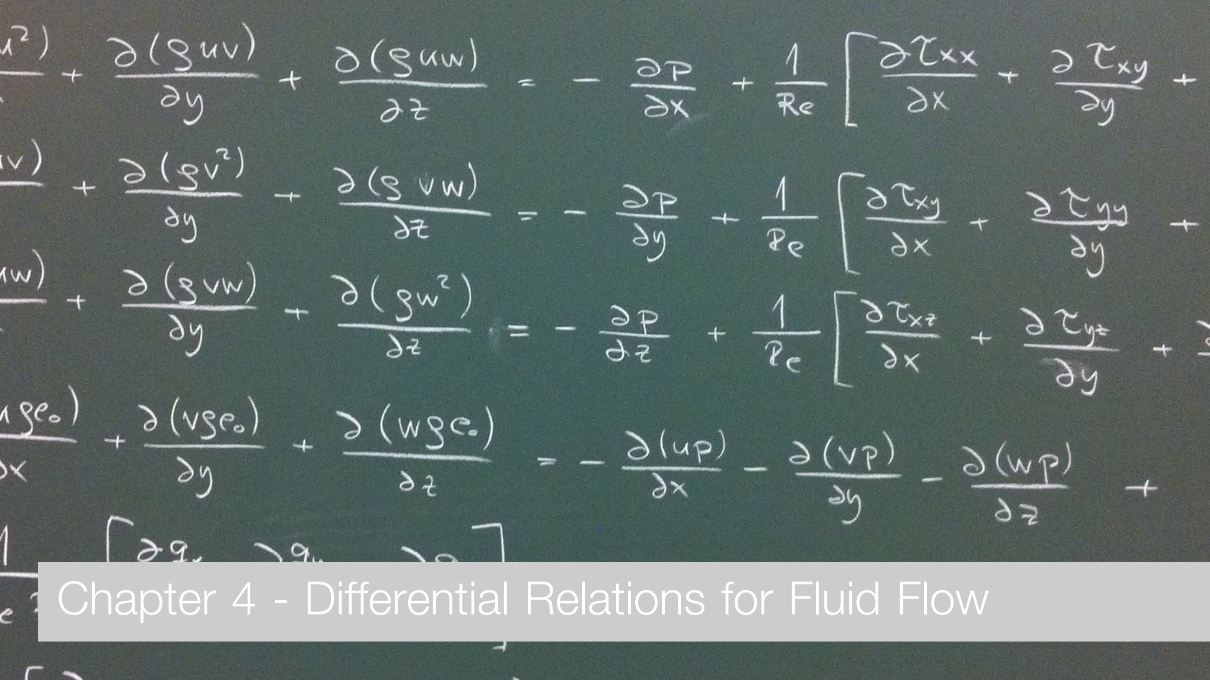


Chapter 4 - Differential Relations for Fluid Flow

Overview

The Pi theorem
non-dimensional equations
rotation
stream function
modeling and similarity
Dimen-sional Analysis
conser-vation relations
Differential Relations
Flow Relations
Bernoulli
Integral Relations
conser-vation relations
hydrostatic forces
buoyancy

Learning Outcomes

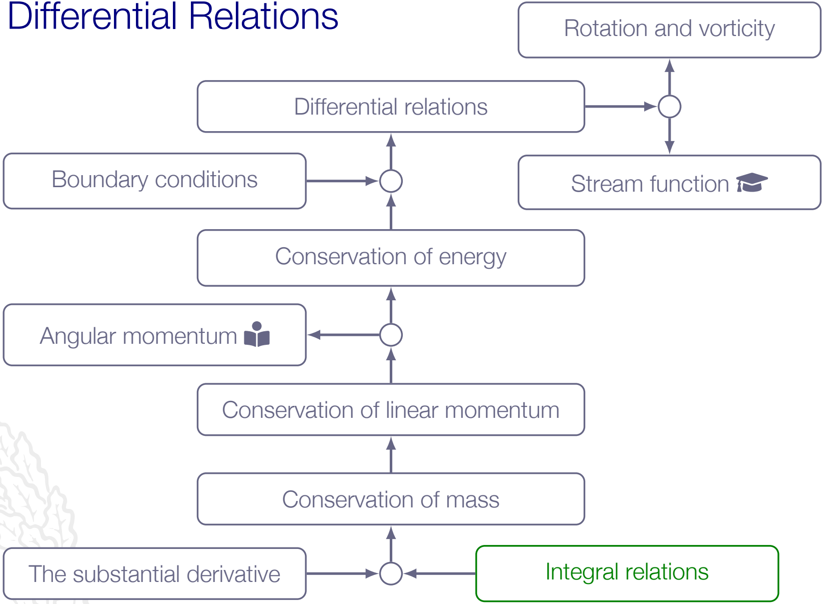
4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

14 **Derive** the continuity, momentum and energy equations on differential form

36 **Define** and explain vorticity

let's push the control volume approach to the limit ...

Roadmap - Differential Relations



Differential Relations

seeking the point-by-point details of a flow pattern by analyzing an infinitesimal region of the flow

Differential Relations

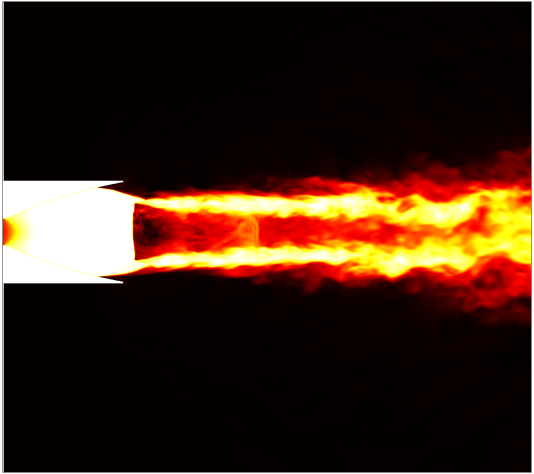
Apply the four basic conservation laws to an infinitesimal control volume

The differential relations are in general very difficult to solve

Analytical solutions exists for a few cases

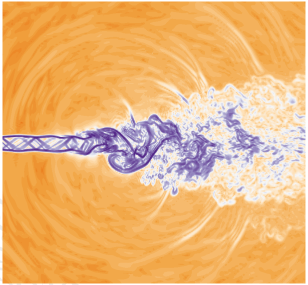
The differential relations form the basis for CFD software

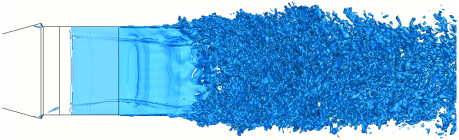
High-Speed Nozzle Flow



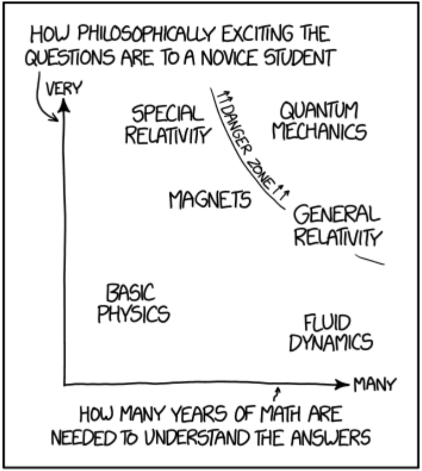
The Acoustic Signature of a Supersonic Jet

Screeching rectangular supersonic jet

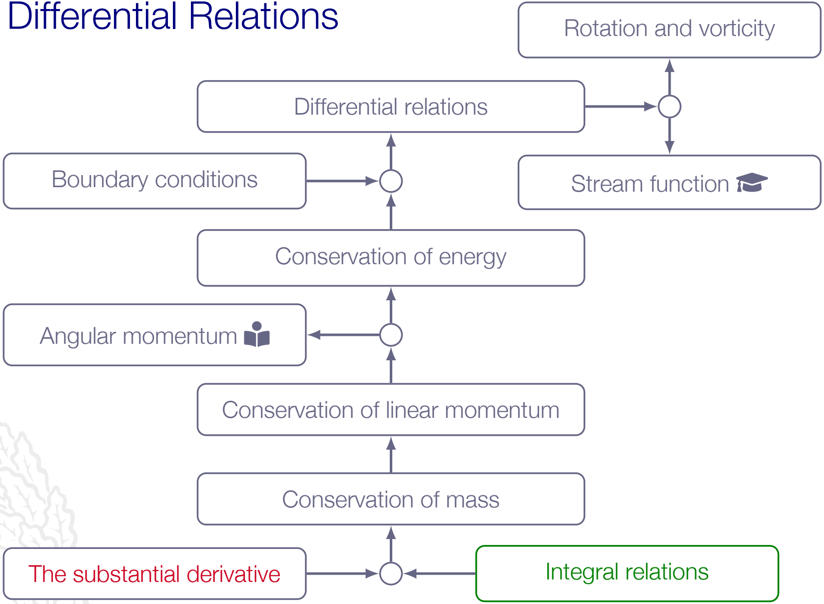




Lots of Equations Ahead



Roadmap - Differential Relations



Frame of Reference

Eulerian: observer fixed in space

Lagrangian: observer follows a fluid particle

recall the speedometer/traffic-camera analogy

Acceleration Field

In order to get to Newton's second law, we need the acceleration vector

Velocity field:

$$\mathbf{V}(\mathbf{r},t) = \mathbf{e}_x u(x,y,z,t) + \mathbf{e}_y v(x,y,z,t) + \mathbf{e}_z w(x,y,z,t)$$

Acceleration field:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{e}_x \frac{du}{dt} + \mathbf{e}_y \frac{dv}{dt} + \mathbf{e}_z \frac{dw}{dt}$$

Acceleration Field

Each scalar component of the velocity vector (u, v, w) is a function of four variables (x, y, z, t) and thus

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

By definition $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$

$$\frac{du(x,y,z,t)}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

Acceleration Field

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V}\cdot\nabla)v$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w$$

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{D\mathbf{V}}{Dt}$$

The Substantial derivative

$$\mathbf{a} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{local} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{convective} = \underbrace{\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}}_{substantial\ derivative} = \frac{D\mathbf{V}}{Dt}$$

local acceleration $(\partial \mathbf{V}/\partial t)$**:**
only non-zero in unsteady flows (always zero in steady-state flows)

convective acceleration $((\mathbf{V}\cdot\nabla)\mathbf{V})$**:**
non-zero for fluid particles that moves through regions of spatially varying velocity

the substantial derivative $(D\mathbf{V}/Dt)$**:**
operator that combines local and convective acceleration

The Substantial derivative

$$\frac{D\varphi}{Dt}$$

the sum of the **local** derivative and the **convective** derivative

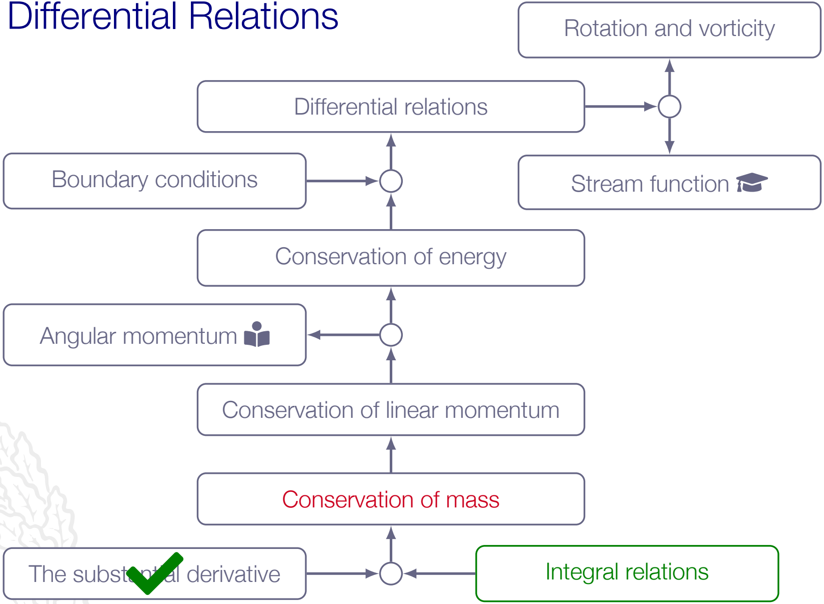
follows a fluid particle but is expressed in an **Eulerian frame of reference**

an **operator** that can be applied to any variable φ

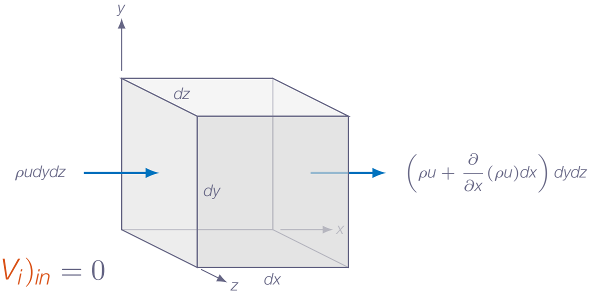
example: the substantial deriviative applied to pressure

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V} \cdot \nabla)p$$

Roadmap - Differential Relations



Mass Conservation



Integral form:

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

Infinitesimal control volume:

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dxdydz$$

$$\sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} \approx \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] dxdydz = 0$$

Mass Conservation

The result is the **continuity equation** - conservation of mass for an infinitesimal control volume

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

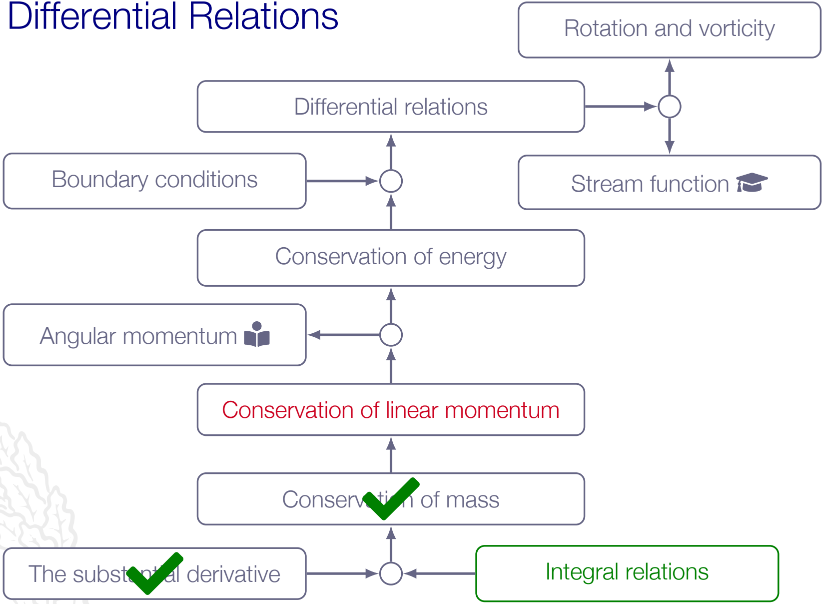
or in more compact form using vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

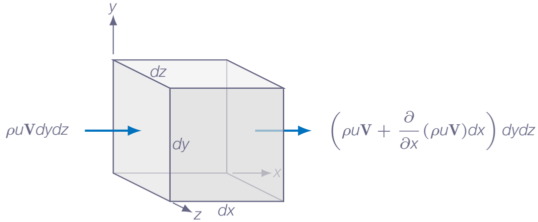
Incompressible flow (constant density)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} = 0$$

Roadmap - Differential Relations



Linear Momentum



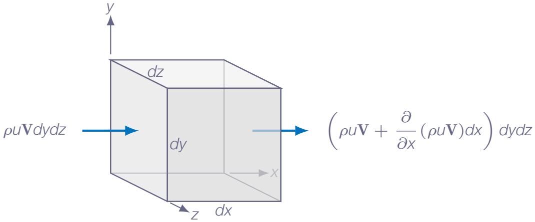
Integral form:

$$\sum \mathbf{F} = \int_{CV} \frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} + \sum(\dot{m}_i \mathbf{V}_i)_{out} - \sum(\dot{m}_i \mathbf{V}_i)_{in}$$

Infinitesimal control volume:

$$\frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} \approx \frac{\partial}{\partial t}(\mathbf{V}\rho)dxdydz$$

Linear Momentum



Face	Inlet momentum flux	Outlet momentum flux
constant x	$\rho u \mathbf{V} dydz$	$\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V})dx\right] dydz$
constant y	$\rho v \mathbf{V} dxdz$	$\left[\rho v \mathbf{V} + \frac{\partial}{\partial y}(\rho v \mathbf{V})dy\right] dxdz$
constant z	$\rho w \mathbf{V} dxdy$	$\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V})dz\right] dxdy$

Linear Momentum

Integral form:

$$\sum \mathbf{F} = \int_{CV} \frac{\partial}{\partial t}(\mathbf{V}\rho)d\mathcal{V} + \sum(\dot{m}_i\mathbf{V}_i)_{out} - \sum(\dot{m}_i\mathbf{V}_i)_{in}$$

Infinitesimal control volume:

$$\sum \mathbf{F} = \frac{\partial}{\partial t}(\mathbf{V}\rho)dxdydz + \left[\frac{\partial}{\partial x}(\rho u\mathbf{V}) + \frac{\partial}{\partial y}(\rho v\mathbf{V}) + \frac{\partial}{\partial z}(\rho w\mathbf{V})\right]dxdydz$$

Linear Momentum

$$\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\rho)}_{\mathbf{V}\frac{\partial \rho}{\partial t}+\rho\frac{\partial \mathbf{V}}{\partial t}} + \underbrace{\frac{\partial}{\partial x}(\rho u\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial x}(\rho u)+\rho u\frac{\partial \mathbf{V}}{\partial x}} + \underbrace{\frac{\partial}{\partial y}(\rho v\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial y}(\rho v)+\rho v\frac{\partial \mathbf{V}}{\partial y}} + \underbrace{\frac{\partial}{\partial z}(\rho w\mathbf{V})}_{\mathbf{V}\frac{\partial}{\partial z}(\rho w)+\rho w\frac{\partial \mathbf{V}}{\partial z}}$$

can be rewritten as

$$\underbrace{\mathbf{V}\left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{V})\right]}_{\text{continuity equation}} + \rho\underbrace{\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\frac{\partial \mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}=\frac{D\mathbf{V}}{Dt}}$$

and thus

$$\sum \mathbf{F} = \rho\frac{D\mathbf{V}}{Dt}dxdydz$$

Linear Momentum - Forces

$$\sum \mathbf{F} = \rho \frac{D\mathbf{V}}{Dt} dxdydz$$

The net force equals mass times acceleration where the acceleration is obtained using the **substantial derivative** - the sum of local and convective acceleration

$\sum \mathbf{F}$:

body forces: gravity and other field forces

surface forces: pressure and viscous stresses

Linear Momentum - Gravity Force

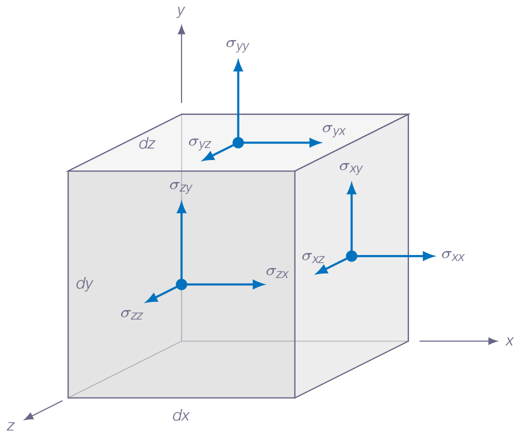
$$d\mathbf{F}_{gravity} = \rho\mathbf{g}dxdydz$$

if gravity is aligned with the negative z-direction

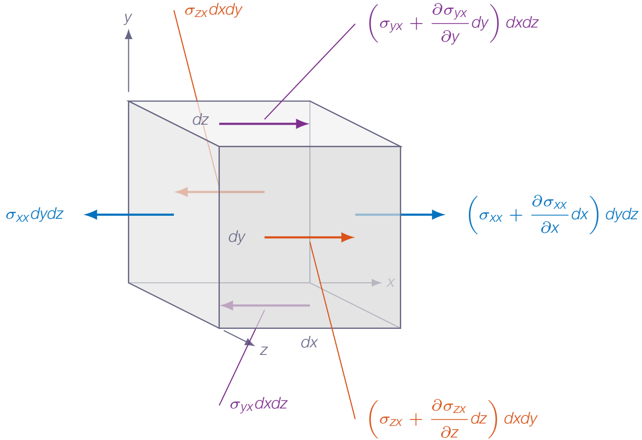
$$d\mathbf{F}_{gravity} = -\mathbf{e}_z\rho g dxdydz$$

Linear Momentum - Surface Forces

$$\sigma_{ij} = \left| \begin{array}{ccc} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{array} \right|$$



Linear Momentum - Surface Forces



$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dxdydz$$

$$\sigma_{xx} = \tau_{xx} - p, \ \sigma_{yx} = \tau_{yx}, \ \sigma_{zx} = \tau_{zx}$$

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right] dxdydz$$

Linear Momentum - Surface Forces

$$dF_{x,surf} = \left[-\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right] dxdydz$$

$$dF_{y,surf} = \left[-\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})\right] dxdydz$$

$$dF_{z,surf} = \left[-\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})\right] dxdydz$$

Linear Momentum - Surface Forces

$$d\mathbf{F} = \left[-\nabla p + \nabla \cdot \tau_{ij}\right] dxdydz$$

where

$$\tau_{ij} = \left|\begin{array}{ccc} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{array}\right|$$

is the viscous stress tensor

Linear Momentum - Forces

Now, inserting the forces into the momentum equation gives

$$\rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij} = \rho \frac{D\mathbf{V}}{Dt}$$

Linear Momentum

vector notation is powerful, tensor notation is even better ...

$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Note! the convective term (RHS) is nonlinear

Linear Momentum - Viscous Stresses in a Newtonian Fluid

Recall:

"For a Newtonian fluid, the viscous stresses are proportional to the element strain and the viscosity"

For incompressible flow:

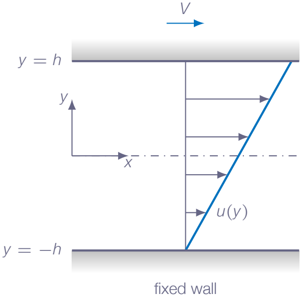
$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x},\ \tau_{yy} = 2\mu\frac{\partial v}{\partial y},\ \tau_{zz} = 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),\ \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

Linear Momentum - Shear Stress Components in a Couette Flow

Couette flow: $u = u(y)$



$$\tau_{xx} = 2\mu\frac{\partial u}{\partial x} = 0$$

$$\tau_{yy} = 2\mu\frac{\partial v}{\partial y} = 0$$

$$\tau_{zz} = 2\mu\frac{\partial w}{\partial z} = 0$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu\frac{\partial u}{\partial y}$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0$$

The incompressible Navier-Stokes equations

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Linear Momentum

The incompressible Navier-Stokes equations

- Non-linear equations
- Three equations and four unknowns (p, u, v, w)
- Combined with the continuity equations we have four equations and four unknowns

ABOUT PROGRAMS MILLENNIUM PROBLEMS PEOPLE PUBLICATIONS EVENTS EUCLID

Millennium Problems

Yang–Mills and Mass Gap

Experiment and computer simulations suggest the existence of a "mass gap" in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

Riemann Hypothesis

The prime number theorem determines the average distribution of the primes. The Riemann hypothesis tells us about the deviation from the average. Formulated in Riemann's 1859 paper, it asserts that all the 'non-obvious' zeros of the zeta function are complex numbers with real part 1/2.

P vs NP Problem

If it is easy to check that a solution to a problem is correct, is it also easy to solve the problem? This is the essence of the P vs NP question. Typical of the NP problems is that of the Hamiltonian Path Problem: given N cities to visit, how can one do this without visiting a city twice? If you give me a solution, I can easily check that it is correct. But I cannot so easily find a solution.

Navier–Stokes Equation

This is the equation which governs the flow of fluids such as water and air. However, there is no proof for the most basic questions one can ask: do solutions exist, and are they unique? Why ask for a proof? Because a proof gives not only certitude, but also understanding.

Hodge Conjecture

The answer to this conjecture determines how much of the topology of the solution set of a system of algebraic equations can be defined in terms of further algebraic equations. The Hodge conjecture is known in certain special cases, e.g., when the solution set has dimension less than four. But in dimension four it is unknown.

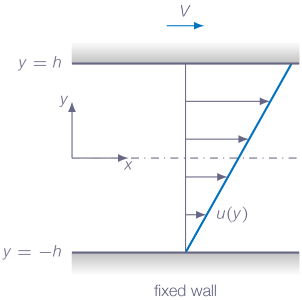
Poincaré Conjecture

In 1904 the French mathematician Henri Poincaré asked if the three dimensional sphere is characterized as the unique simply connected three manifold. This question, the Poincaré conjecture, was a special case of Thurston's geometrization conjecture. Perelman's proof tells us that every three manifold is built from a set of standard pieces, each with one of eight well-understood geometries.

Birch and Swinnerton-Dyer Conjecture

Supported by much experimental evidence, this conjecture relates the number of points on an elliptic curve mod p to the rank of the group of rational points. Elliptic curves, defined by cubic equations in two variables, are fundamental mathematical objects that arise in many areas: Wiles' proof of the Fermat Conjecture, factorization of numbers into primes, and cryptography, to name three.

Example - Couette Flow



Assumptions:

1. incompressible ($\rho = const$)
2. steady-state
3. lower plate fixed, upper plate moving with the velocity V
4. flow only in the x-direction $v = w = 0,\ u \neq 0$
5. no pressure gradient

Example - Couette Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \Rightarrow \{v = w = 0,\ \frac{\partial p}{\partial x} = 0\} \Rightarrow \mu\frac{\partial^2 u}{\partial y^2} = 0$$

Example - Couette Flow

$$\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u = ay + b$$

boundary conditions:

$$\left.\begin{array}{l} u(h) = V \\ u(-h) = 0 \end{array}\right\} \Rightarrow$$

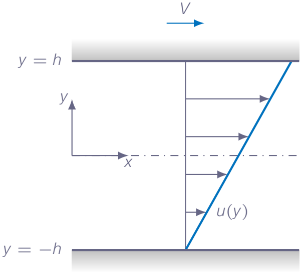
$$\begin{array}{rcrcr} & V & = & ah & + & b \\ + & 0 & = & -ah & + & b \\ \hline & V & = & & & 2b \end{array}$$

$$b = \frac{V}{2}$$

$$\begin{array}{rcrcr} & V & = & ah & + & b \\ - & 0 & = & -ah & + & b \\ \hline & V & = & 2ah & & \end{array}$$

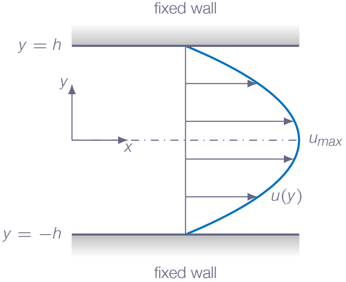
$$a = \frac{V}{2h}$$

Example - Couette Flow



$$u = \frac{V}{2}\left(\frac{y}{h} - 1\right)$$

Example - Poiseuille Flow



Assumptions:

1. incompressible ($\rho = const$)
2. steady-state
3. lower and upper plate fixed
4. flow only in the x-direction $v = w = 0, \; u \neq 0$
5. pressure gradient driven

Example - Poiseuille Flow

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \{v = w = 0\} \Rightarrow \frac{\partial u}{\partial x} = 0$$

momentum equation (x-direction):

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \Rightarrow \{v = w = 0\} \Rightarrow \mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$$

Example - Poiseuille Flow

momentum equation (y-direction and z-direction):

$$\left.\begin{aligned} \frac{\partial p}{\partial y} &= 0 \\ \frac{\partial p}{\partial z} &= 0 \end{aligned}\right\} \Rightarrow p = p(x)$$

Example - Poiseuille Flow

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} = const < 0$$

Why constant?

1. RHS function of x only
2. LHS function of y only
3. RHS=LHS $\Rightarrow$ must be a constant

Why < 0?

pressure must decrease in the flow direction

Example - Poiseuille Flow

$$\mu \frac{d^2 u}{dy^2} = \frac{dp}{dx} = const < 0$$

$$u = \frac{1}{2\mu} \frac{dp}{dx} y^2 + ay + b$$

Example - Poiseuille Flow

boundary conditions:

$$\left.\begin{array}{l} u(h) = 0 \\ u(-h) = 0 \end{array}\right\} \Rightarrow$$

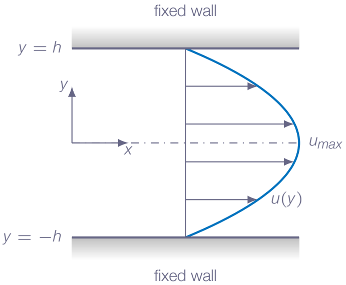
$$\begin{array}{rrcrrrr} & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & + \; ah & + & b \\ + & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & - \; ah & + & b \\ \hline & 0 & = & \frac{1}{\mu}\frac{dp}{dx}h^2 & & + & 2b \end{array}$$

$$b = -\frac{1}{2\mu}\frac{dp}{dx}h^2$$

$$\begin{array}{rrcrrrr} & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & + \; ah & + & b \\ - & 0 & = & \frac{1}{2\mu}\frac{dp}{dx}h^2 & - \; ah & + & b \\ \hline & 0 & = & & \text{2ah} & & \end{array}$$

$$a = 0$$

Example - Poiseuille Flow

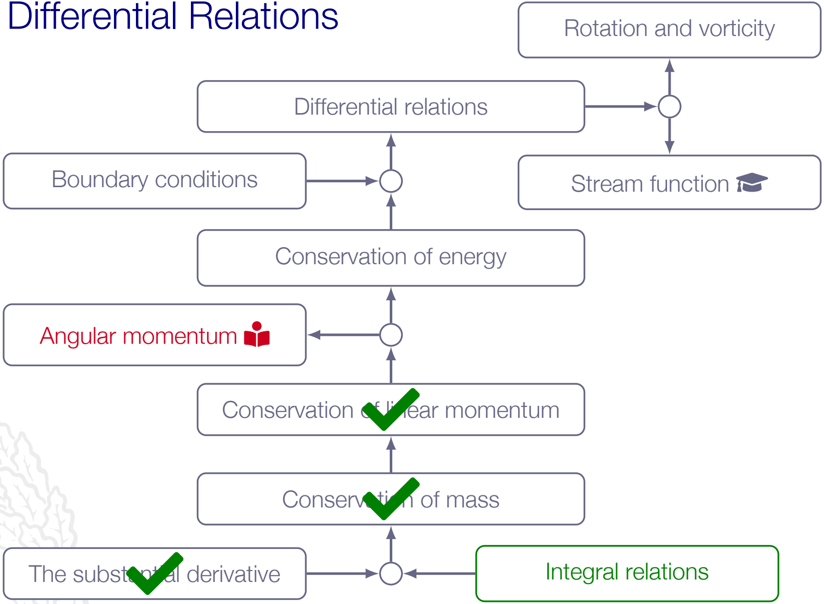


$$u = -\frac{h^2}{2\mu}\frac{dp}{dx}\left(1 - \left(\frac{y}{h}\right)^2\right)$$

$$\frac{du}{dy} = \frac{dp}{dx}\frac{y}{\mu} \Rightarrow \left.\frac{du}{dy}\right|_{y=0} = 0 \Rightarrow u_{max} = u(0) = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

(remember: $dp/dx < 0$)

Roadmap - Differential Relations



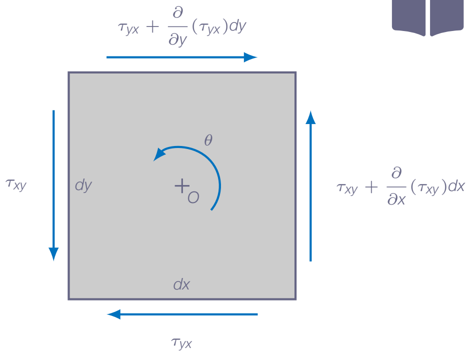
Angular Momentum

Integral form:

$$\sum \mathbf{M}_o = \frac{d}{dt}\left(\int_{CV} \rho(\mathbf{r} \times \mathbf{V}) d\mathcal{V}\right) + \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V} \cdot \mathbf{n}) dA$$

Angular Momentum

Infinitesimal control volume:



1. axis through o parallel to the z-axis
2. axis through the centroid of the element
3. θ angle of rotation about o

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz =$$

$$\frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Angular Momentum

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})dy\right] dxdydz =$$

$$\frac{1}{12}\rho(dxdydz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Neglect higher-order differential terms gives

$$\boxed{\tau_{xy} \approx \tau_{yx}}$$

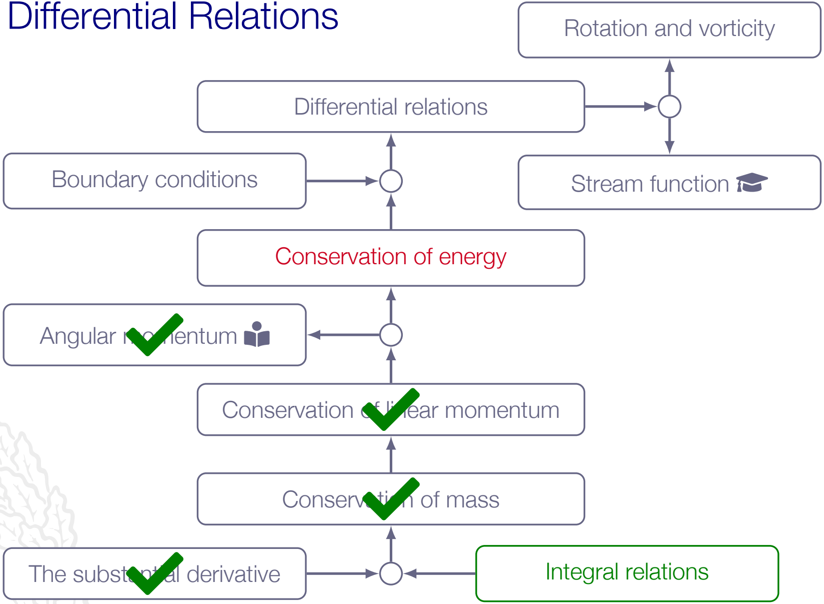
Analogously, we may obtain $\tau_{xz} \approx \tau_{zx}$ and $\tau_{zy} \approx \tau_{yz}$

Angular Momentum

Note! there is no differential angular momentum equation ...

the only result from this section is that shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$

Roadmap - Differential Relations



The Energy Equation

Integral formulation:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

$$h = e + p/\rho$$

Differential form:

$$\dot{Q} - \dot{W}_\nu = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)\right] dxdydz$$

$\dot{W}_s = 0$ we can not have a infinitesimal shaft protruding the control volume

The Energy Equation

$$\dot{Q} - \dot{W}_\nu = \left[\underbrace{\frac{\partial}{\partial t}(\rho e)}_{I} + \underbrace{\frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h)}_{II}\right] dxdydz$$

Part I.

$$\frac{\partial}{\partial t}(\rho e) = e\frac{\partial \rho}{\partial t} + \rho\frac{\partial e}{\partial t}$$

The Energy Equation

Part II.

$$\frac{\partial}{\partial x}(\rho uh) + \frac{\partial}{\partial y}(\rho vh) + \frac{\partial}{\partial z}(\rho wh) =$$

$$\underbrace{\frac{\partial}{\partial x}(\rho ue) + \frac{\partial}{\partial y}(\rho ve) + \frac{\partial}{\partial z}(\rho we)}_{*} + \underbrace{\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp)}_{**}$$

Part II*

$$\frac{\partial}{\partial x}(\rho ue) + \frac{\partial}{\partial y}(\rho ve) + \frac{\partial}{\partial z}(\rho we) =$$

$$e\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] + \rho\left[u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z}\right]$$

The Energy Equation

Part II**

$$\frac{\partial}{\partial x}(up) + \frac{\partial}{\partial y}(vp) + \frac{\partial}{\partial z}(wp) =$$

$$p\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} =$$

$$p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

reassemble and collect terms:

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u h) + \frac{\partial}{\partial y}(\rho v h) + \frac{\partial}{\partial z}(\rho w h) =$$

$$\underbrace{\rho \left[\frac{\partial e}{\partial t} + u \frac{\partial e}{\partial x} + v \frac{\partial e}{\partial y} + w \frac{\partial e}{\partial z} \right]}_{\frac{\partial e}{\partial t} + (\mathbf{V} \cdot \nabla) e = \frac{De}{Dt}} +$$

$$\underbrace{e \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right]}_{continuity\ equation} +$$

$$p \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

$$\dot{Q} - \dot{W}_\nu = \left[\rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p\right] dxdydz$$

The Energy Equation - Added Heat

Now, let's have a look at the added heat term $\dot{Q}$

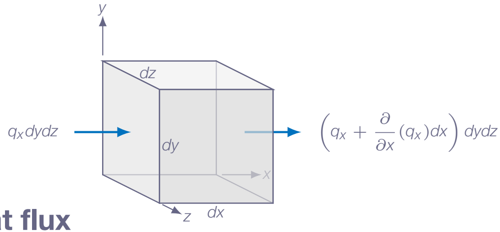
Only **conduction** will be considered (no radiation)

According the **Fourier's law** of conduction, the **heat flux** is proportional to the **temperature gradient**

$$\mathbf{q} = -k\nabla T$$

where k is the **thermal conductivity** and $\mathbf{q}$ is heat transfer per unit area

The Energy Equation - Added Heat



Face	Inlet heat flux	Outlet heat flux
constant x	$q_x dydz$	$\left[q_x + \frac{\partial q_x}{\partial x}dx\right] dydz$ where $q_x = -k\frac{\partial T}{\partial x}$
constant y	$q_y dxdz$	$\left[q_y + \frac{\partial q_y}{\partial y}dy\right] dxdz$ where $q_y = -k\frac{\partial T}{\partial y}$
constant z	$q_z dxdy$	$\left[q_z + \frac{\partial q_z}{\partial z}dz\right] dxdy$ where $q_z = -k\frac{\partial T}{\partial z}$

The Energy Equation - Added Heat

net added heat:

$$\dot{Q} = -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] dxdydz = -\nabla \cdot \mathbf{q} dxdydz$$

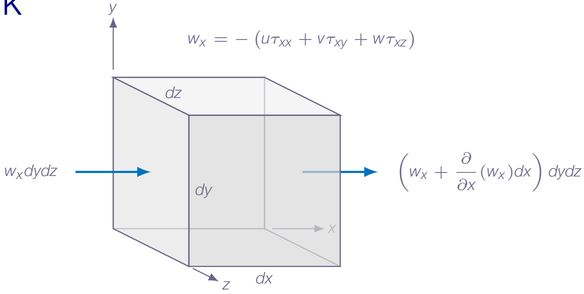
or

$$\dot{Q} = \nabla \cdot (k \nabla T) dxdydz$$

The Energy Equation - Viscous Work

The rate of work done by viscous stresses equals the product of the **stress component**, its corresponding **velocity component** and **surface area**

The Energy Equation - Viscous Work



$$\dot{W}_\nu = -\left[\frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \right.$$

$$\left.\frac{\partial}{\partial y}(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\tau_{zz})\right] dxdydz =$$

$$-\nabla \cdot (\mathbf{V} \cdot \tau_{ij})dxdydz$$

The Energy Equation

with the derived expressions for heat and viscous work we end up with

$$\nabla \cdot (k\nabla T) + \nabla \cdot (\mathbf{V} \cdot \tau_{ij}) = \rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

Now, introducing the viscous-dissipation function ϕ for Newtonian fluids and incompressible flows

$$\nabla \cdot (\mathbf{V} \cdot \tau_{ij}) = \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi$$

where

$$\phi = \mu \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right]$$

The Energy Equation

Note!

”All terms in the viscous-dissipation function are quadratic which means that in a viscous flow there will always be losses, which is in line with the second law of thermodynamics”

The Energy Equation

$$\nabla \cdot (k\nabla T) + \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi = \rho \frac{De}{Dt} + p\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla p$$

Now, let's eliminate the term $\mathbf{V} \cdot (\nabla \cdot \tau_{ij})$ in the energy equation:

Momentum equation:

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Multiply the momentum equation with the velocity vector (scalar product)

$$\mathbf{V} \cdot (\nabla \cdot \tau_{ij}) = \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p$$

The Energy Equation

Energy equation:

$$\rho \frac{De}{Dt} + \mathbf{V} \cdot \nabla p + p \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \mathbf{V} \cdot (\nabla \cdot \tau_{ij}) + \phi$$

eliminate $\mathbf{V} \cdot (\nabla \cdot \tau_{ij})$ using the result from previous slide

$$\rho \frac{De}{Dt} + \mathbf{V} \cdot \nabla p + p \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + \phi$$

Doesn't seem like a very wise move at this stage ...

The Energy Equation

As the next step, express energy per unit mass (e) as the sum of internal energy, kinetic energy, and potential energy (*as we did in Chapter 3*)

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

or in vector form:

$$e = \hat{u} + \frac{1}{2}\mathbf{V} \cdot \mathbf{V} - \mathbf{g}\mathbf{r}$$

where $\mathbf{g} = -(g_x, g_y, g_z)$ is the gravity vector and $\mathbf{r} = (x, y, z)$ is the location vector

The Energy Equation

$$e = \hat{u} + \frac{1}{2}\mathbf{V}\cdot\mathbf{V} - \mathbf{g}\mathbf{r}$$

Now, apply the substantial derivative to e

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \underbrace{\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V})}_{=\mathbf{V}\cdot\frac{D\mathbf{V}}{Dt}*} - \underbrace{\frac{D}{Dt}(\mathbf{g}\mathbf{r})}_{=\mathbf{V}\cdot\mathbf{g}*} = \frac{D\hat{u}}{Dt} + \mathbf{V}\cdot\frac{D\mathbf{V}}{Dt} - \mathbf{V}\cdot\mathbf{g}$$

* *details on the following slides*

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[\underbrace{\frac{\partial}{\partial t}(\mathbf{V}\cdot\mathbf{V})}_{I} + \underbrace{(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V})}_{II}\right]$$

I:

$$\frac{\partial}{\partial t}(\mathbf{V}\cdot\mathbf{V}) = \frac{\partial}{\partial t}(u^2+v^2+w^2) = \frac{\partial u^2}{\partial t} + \frac{\partial v^2}{\partial t} + \frac{\partial w^2}{\partial t} =$$

$$2u\frac{\partial u}{\partial t} + 2v\frac{\partial v}{\partial t} + 2w\frac{\partial w}{\partial t} = 2\mathbf{V}\cdot\frac{\partial \mathbf{V}}{\partial t}$$

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[2\mathbf{V}\cdot\frac{\partial\mathbf{V}}{\partial t} + \underbrace{(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V})}_{II}\right]$$

II:

$$(\mathbf{V}\cdot\nabla)(\mathbf{V}\cdot\mathbf{V}) = (\mathbf{V}\cdot\nabla)(u^2+v^2+w^2) =$$

$$= u\frac{\partial u^2}{\partial x} + u\frac{\partial v^2}{\partial x} + u\frac{\partial w^2}{\partial x} + v\frac{\partial u^2}{\partial x} + v\frac{\partial v^2}{\partial x} + v\frac{\partial w^2}{\partial x} + w\frac{\partial u^2}{\partial x} + w\frac{\partial v^2}{\partial x} + w\frac{\partial w^2}{\partial x} =$$

$$= 2\left[u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} + uw\frac{\partial w}{\partial x} + uv\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial x} + vw\frac{\partial w}{\partial x} + uw\frac{\partial u}{\partial x} + vw\frac{\partial v}{\partial x} + w^2\frac{\partial w}{\partial x}\right] =$$

$$= 2\mathbf{V}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}$$

The Energy Equation

$$\frac{1}{2}\frac{D}{Dt}(\mathbf{V}\cdot\mathbf{V}) = \frac{1}{2}\left[2\mathbf{V}\cdot\frac{\partial\mathbf{V}}{\partial t} + 2\mathbf{V}\cdot(\mathbf{V}\cdot\nabla)\mathbf{V}\right] = \mathbf{V}\cdot\frac{D\mathbf{V}}{Dt}$$

The Energy Equation

$$\frac{D}{Dt}(\mathbf{gr}) = \frac{\partial}{\partial t}(\mathbf{gr}) + (\mathbf{V}\cdot\nabla)(\mathbf{gr}) = \underbrace{\frac{\partial}{\partial t}(g_x x,\ g_y y,\ g_z z)}_{=(0,0,0)} + (\mathbf{V}\cdot\nabla)(g_x x,\ g_y y,\ g_z z) =$$

$$\underbrace{\mathbf{V}\cdot\left(x\frac{\partial g_x}{\partial x} + g_x\frac{\partial x}{\partial x},\ y\frac{\partial g_y}{\partial y} + g_y\frac{\partial y}{\partial y},\ z\frac{\partial g_z}{\partial z} + g_z\frac{\partial z}{\partial z}\right)}_{\frac{\partial g_x}{\partial x}=\frac{\partial g_y}{\partial y}=\frac{\partial g_z}{\partial z}=0 \text{ and } \frac{\partial x}{\partial x}=\frac{\partial y}{\partial y}=\frac{\partial z}{\partial z}=1} = \mathbf{V}\cdot(g_x,\ g_y,\ g_z) = \mathbf{V}\cdot\mathbf{g}$$

The Energy Equation

Now, insert

$$\frac{De}{Dt} = \frac{D\hat{u}}{Dt} + \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \mathbf{V} \cdot \mathbf{g}$$

in the energy equation

$$\rho \frac{D\hat{u}}{Dt} + \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + p \nabla \cdot \mathbf{V} =$$

$$\nabla \cdot (k \nabla T) + \rho \mathbf{V} \cdot \frac{D\mathbf{V}}{Dt} - \rho \mathbf{V} \cdot \mathbf{g} + \mathbf{V} \cdot \nabla p + \phi$$

The highlighted terms cancel each other

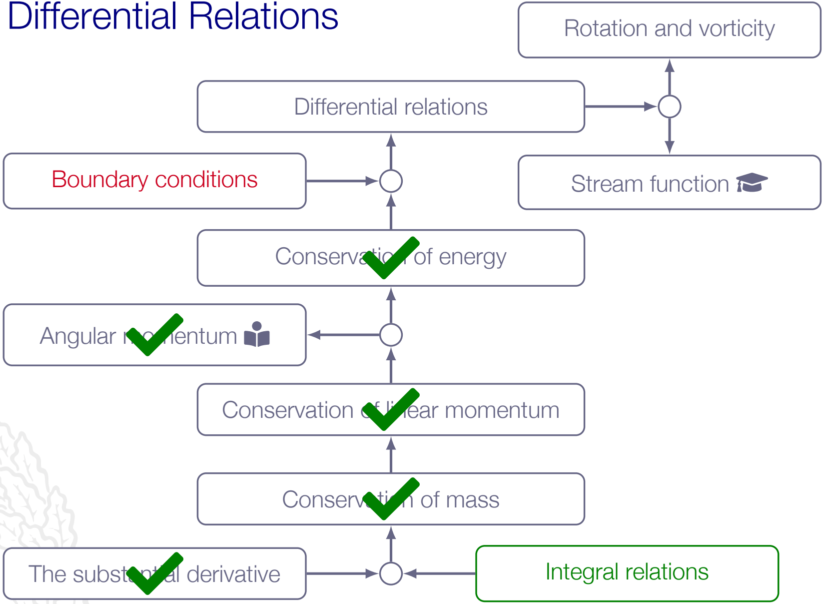
Ok, this was why momentum equation was used here ...

The Energy Equation

$$\rho \frac{D\hat{u}}{Dt} + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\nabla T) + \phi$$

Local and convective changes of internal energy are balanced by pressure work, heat addition and viscous dissipation – viscous dissipation will always increase the internal energy of the fluid

Roadmap - Differential Relations



Flow Equations on Differential Form

Continuity: $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

Momentum: $$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \tau_{ij}$$

Energy: $$\rho \frac{D\hat{u}}{Dt} + p \nabla \cdot \mathbf{V} = \nabla \cdot (k \nabla T) + \phi$$

five equations and seven unknowns $(\rho, u, v, w, p, \hat{u}, T) \Rightarrow$ two additional relations needed:

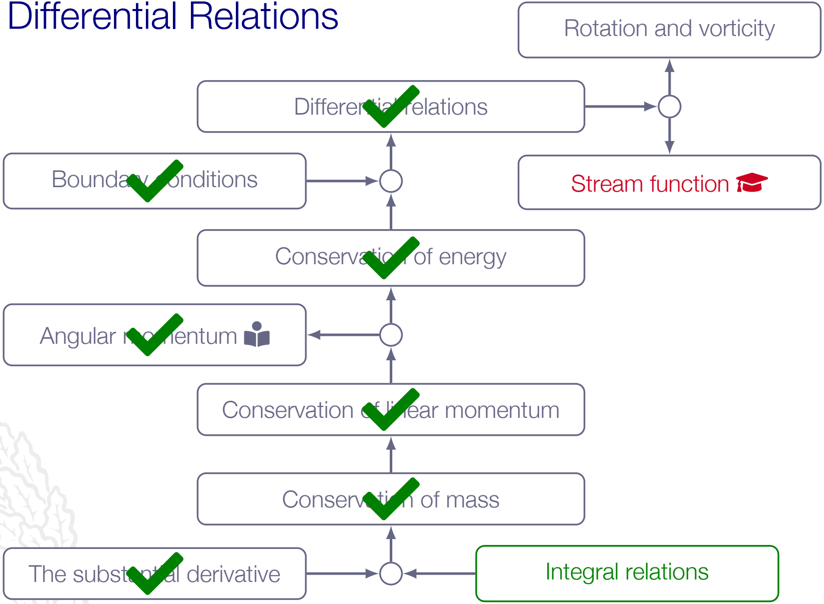
$$\rho = \rho(p, T), \; \hat{u} = \hat{u}(p, T)$$

Flow Equations on Differential Form

Boundary conditions:

1. solid wall: no slip, no temperature jump
2. inlet, outlet (massflow, pressure, total flow conditions, flow angle)
3. liquid-gas interface

Roadmap - Differential Relations



The Stream Function (*for the interested*)

fulfill the continuity equation and solve the momentum equation directly for the single variable ψ

The Stream Function (*for the interested*)

incompressible, two-dimensional flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

define $\psi(x, y)$ such that

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

and thus

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x}$$

or

$$\mathbf{V} = \left[\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right]$$

The Stream Function (*for the interested*)

The rotation of the flow field is calculated using the curl operator

$$\mathit{curl}(\mathbf{V}) = \nabla \times \mathbf{V} = -\nabla^2 \psi \mathbf{e}_z$$

The Stream Function (*for the interested*)

Now, apply the curl operator to the momentum equation

$$\nabla \times \frac{D\mathbf{V}}{Dt} = \underbrace{\nabla \times \mathbf{g}}_{=0} - \frac{1}{\rho}\underbrace{\nabla \times \nabla p}_{=0} + \nu\nabla \times \nabla^2\mathbf{V} = \nu\nabla \times \nabla^2\mathbf{V}$$

$$\nabla \times \frac{\partial \mathbf{V}}{\partial t} + \nabla \times (\mathbf{V}\cdot\nabla)\mathbf{V} = \nu\nabla \times \nabla^2\mathbf{V}$$

$$\left.\begin{array}{l} \dfrac{\partial \mathbf{V}}{\partial t} = 0 \ (\textit{steady}) \\[2ex] \nu\nabla \times \nabla^2\mathbf{V} = \nu\nabla^2(\nabla \times \mathbf{V}) \end{array}\right\} \Rightarrow \nabla \times (\mathbf{V}\cdot\nabla)\mathbf{V} = \nu\nabla^2(\nabla \times \mathbf{V})$$

The Stream Function (*for the interested*)

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\mathbf{V}\cdot\mathbf{V}) - \mathbf{V}\times(\nabla\times\mathbf{V}) = \nabla\left(\frac{V^2}{2}\right) - \mathbf{V}\times(\nabla\times\mathbf{V})$$

and thus

$$\nabla\times(\mathbf{V}\cdot\nabla)\mathbf{V} = \underbrace{\nabla\times\nabla\left(\frac{V^2}{2}\right)}_{=0} - \nabla\times\mathbf{V}\times(\nabla\times\mathbf{V}) = \nabla\times(\nabla\times\mathbf{V})\times\mathbf{V}$$

$$\nabla\times(\nabla\times\mathbf{V})\times\mathbf{V} =$$

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V}) - ((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V} + \underbrace{(\nabla\times\mathbf{V})(\nabla\cdot\mathbf{V})}_{=0\ (\textit{incompressible})} + \mathbf{V}\underbrace{(\nabla\cdot(\nabla\times\mathbf{V}))}_{=0} =$$

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V}) - ((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}$$

The Stream Function (*for the interested*)

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V})-((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}=\nu\nabla^2(\nabla\times\mathbf{V})$$

insert the stream function

$$(\mathbf{V}\cdot\nabla)(\nabla\times\mathbf{V})=(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x},0)\cdot(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})(0,0,-\nabla^2\psi)$$

$$((\nabla\times\mathbf{V})\cdot\nabla)\mathbf{V}=(0,0,-\nabla^2\psi)\cdot(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x},0)=0$$

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi)=\nu\nabla^2(\nabla^2\psi)$$

Stream Function (*for the interested*)

$$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi)$$

+ one equation for ψ that fulfills both the momentum and continuity equations
+ scalar equation
- contains fourth-order derivatives

Stream Function (*for the interested*)

Definition of a streamline in two dimensions:

$$\frac{dx}{u} = \frac{dy}{v}$$

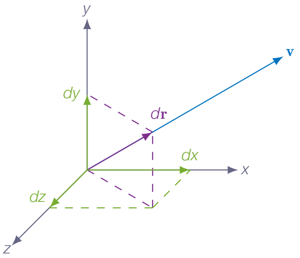
or

$$udy - vdx = 0$$

and thus

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = d\psi = 0$$

or ψ is **constant along a streamline** ...



$d\mathbf{r}$ is aligned with $\mathbf{v}$

$dx \propto u$

$dy \propto v$

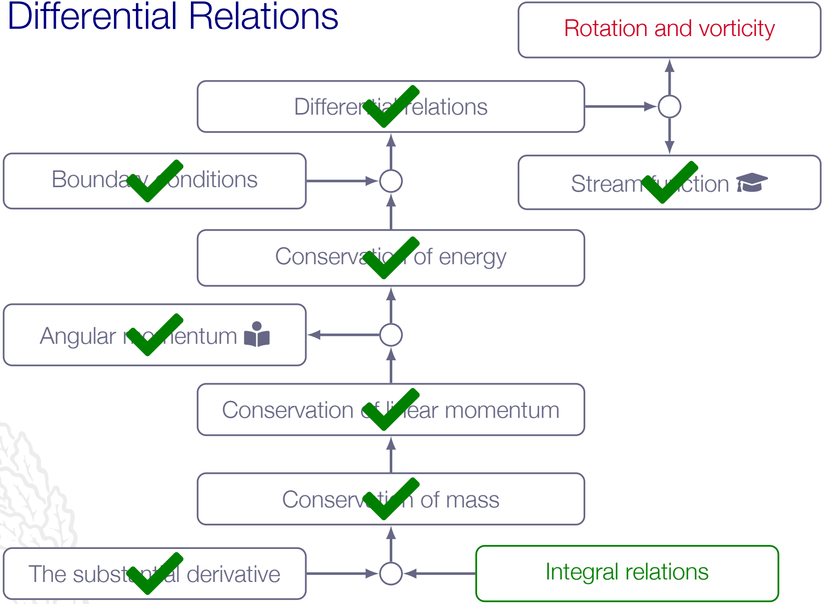
$dz \propto w$

Stream Function (*for the interested*)

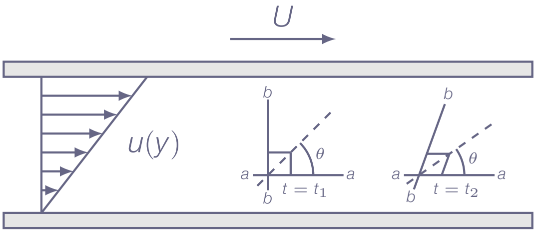
Implication:

1. Lines of constant ψ are streamlines of the flow
2. If we know $\psi(x, y)$, lines of constant ψ will be streamlines of the flow

Roadmap - Differential Relations



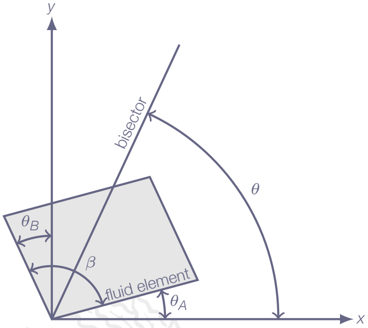
Flow Rotation



Is the Couette flow irrotational?

Hint: note the change of the fluid element bisector angle θ over time

Flow Rotation - Dynamics of a Fluid Element Bisector Angle



$$\beta = \frac{\pi}{2} + \theta_B - \theta_A$$

$$\theta = \frac{\beta}{2} + \theta_A = \frac{\pi}{4} + \frac{1}{2}(\theta_A + \theta_B)$$

the angular velocity of the bisector:

$$\dot{\theta} = \frac{1}{2}\left(\dot{\theta}_A + \dot{\theta}_B\right)$$

Flow Rotation - Fluid Element Deformation

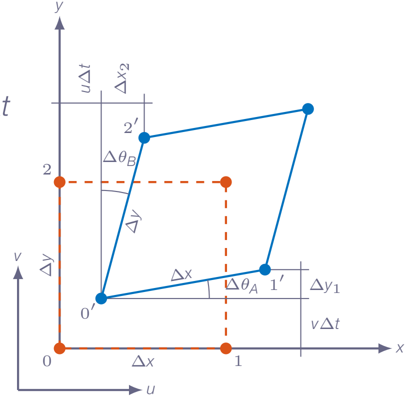
$$\sin(\Delta\theta_A) = \frac{\Delta y_1}{\Delta x} \approx \frac{\left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta t - v\Delta t}{\Delta x} = \frac{\partial v}{\partial x}\Delta t$$

$\sin(\Delta\theta_A) \approx \Delta\theta_A$ for small angles

$$\Rightarrow \underbrace{\frac{\Delta\theta_A}{\Delta t}}_{=\dot{\theta}_A} \approx \frac{\partial v}{\partial x}$$

in the same way $\dot{\theta}_B \approx -\dfrac{\partial u}{\partial y}$

the angular velocity of the bisector: $\dot{\theta} = \frac{1}{2}\left(\dot{\theta}_A + \dot{\theta}_B\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$



Flow Rotation

From previous slide we get the angular velocity about the z axis

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Using the same reasoning, we can get the angular velocities about the x and y axes

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Flow Rotation

$$\omega = \frac{1}{2}\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \frac{1}{2}curl(\mathbf{V})$$

The flow **vorticity** ζ is defined as:

$$\zeta = 2\omega = curl(\mathbf{V})$$

Flows with **zero vorticity** are called **irrotational**

Frictionless Flow

For **frictionless flow**, the momentum equation reduces to Euler's equation

$$\rho \frac{D\mathbf{V}}{Dt} = \rho \mathbf{g} - \nabla p$$

The acceleration term on the left-hand side

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}$$

can be rewritten using the following vector identity

$$(\mathbf{V} \cdot \nabla)\mathbf{V} \equiv \nabla(\frac{1}{2}V^2) + \zeta \times \mathbf{V}$$

Doesn't seem like a simplification but let's try ...

Frictionless Flow

1. combine Euler's equation with the modified acceleration term
2. divide by ρ
3. dot product between the entire equation and an arbitrary displacement vector $d\mathbf{r}$

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla(\frac{1}{2}V^2) + \zeta \times \mathbf{V} + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

Frictionless Flow

Now, for reasons that will be obvious later, we want to get rid of the term $(\zeta \times \mathbf{V}) \cdot d\mathbf{r}$

We have four alternative ways to accomplish that ...

1. $\mathbf{V} = 0$
   no flow $\Rightarrow$ not interesting

2. $\zeta = 0$
   irrotational flow

3. $d\mathbf{r}$ perpendicular to $(\zeta \times \mathbf{V})$
   just strange

4. $d\mathbf{r}$ parallel to $\mathbf{V}$
   flow along a streamline

Frictionless Flow

Let's go for the fourth alternative - **flow along a streamline**:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla(\frac{1}{2}V^2) + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

performing the scalar products gives

$$-\mathbf{g} \cdot d\mathbf{r} = \{\mathbf{g} = -g\mathbf{e}_z\} = g dz$$

$$\nabla p \cdot d\mathbf{r} = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz = dp$$

$$\nabla(\frac{1}{2}V^2) \cdot d\mathbf{r} = \frac{1}{2}\left(\frac{\partial V^2}{\partial x}dx + \frac{\partial V^2}{\partial y}dy + \frac{\partial V^2}{\partial z}dz\right) = \frac{1}{2}d\left(V^2\right)$$

Frictionless Flow

The resulting equation for flow along a streamline becomes:

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + \frac{1}{2} d\left(V^2\right) + \frac{dp}{\rho} + g dz = 0$$

Frictionless Flow

Integrate between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

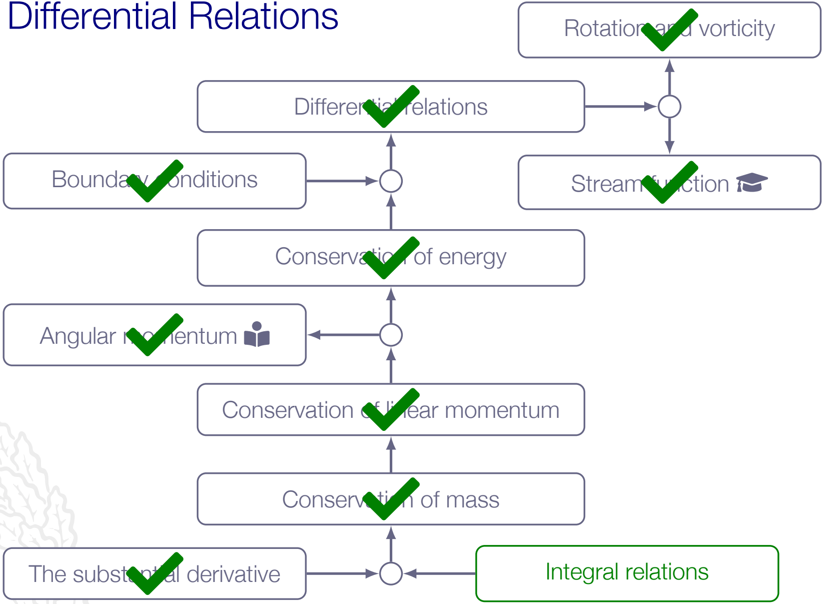
The Bernoulli equation for frictionless unsteady flow

Steady incompressible flow gives

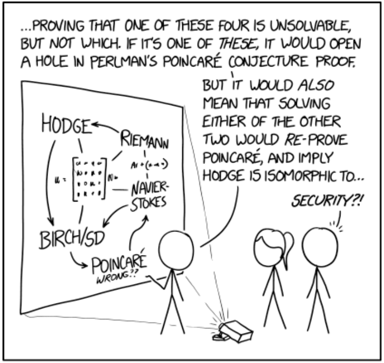
$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = constant$$

Note! for irrotational flow this last results holds in the entire flow field with the same constant

Roadmap - Differential Relations



Millennium Problems



I'M TRYING TO MAKE IT SO THE CLAY MATHEMATICS INSTITUTE HAS TO OFFER AN EIGHTH PRIZE TO WHOEVER FIGURES OUT WHO THEIR OTHER PRIZES SHOULD GO TO.