Fluid Mechanics - MTF053

Chapter 3

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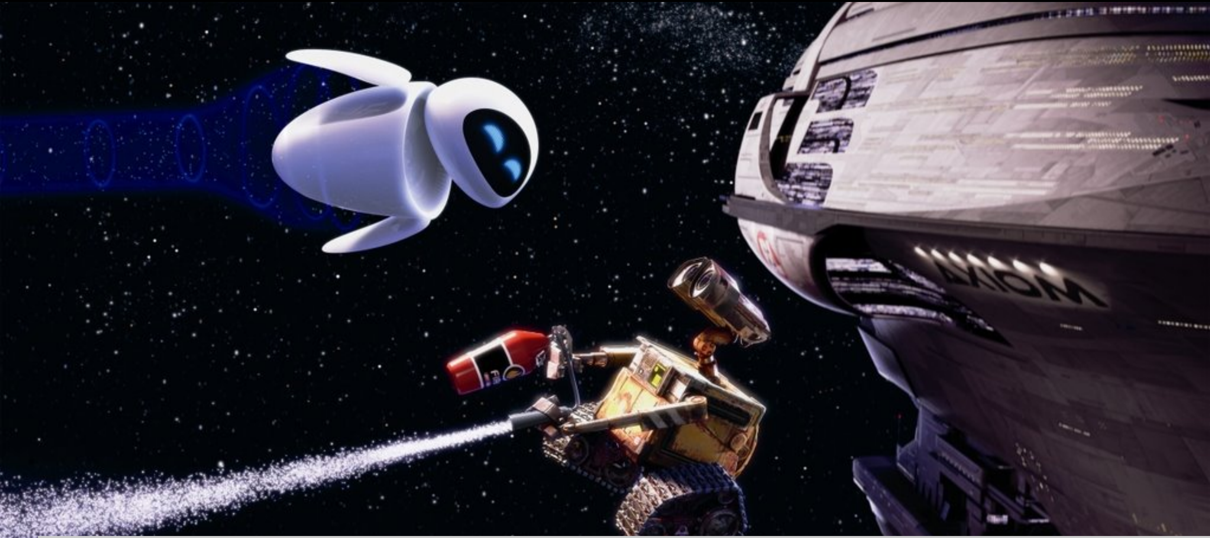


Chapter 3 - Integral Relations for a Control Volume

Overview


conser-vation relations
Differential Relations
sional Analysis
Bernoulli
Flow Relations
Integral Relations
conser-vation relations
Reynolds transport theorem
buoyancy
hydrostatic forces
Pressure
velocity field
Reynolds number

Learning Outcomes

4 Be **able to categorize** a flow and **have knowledge about** how to select applicable methods for the analysis of a specific flow based on category

12 **Define** Reynolds transport theorem using the concepts control volume and system

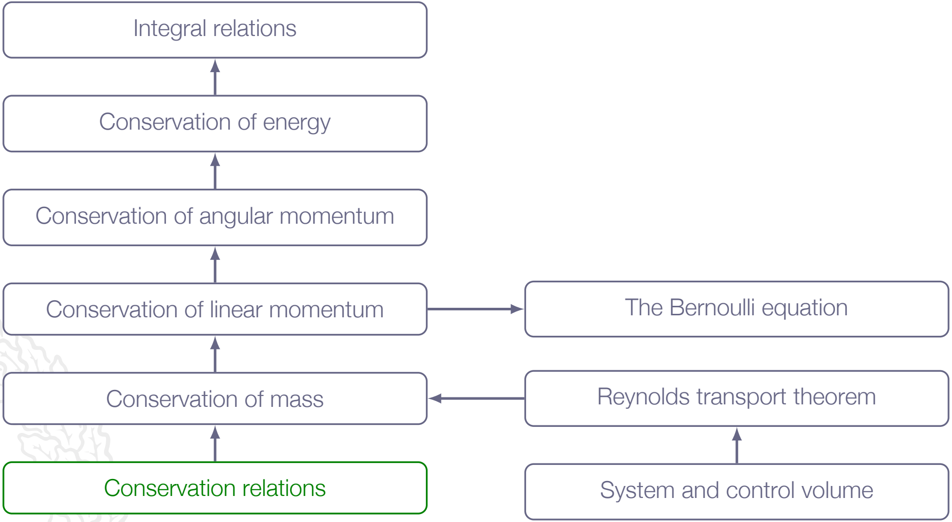
13 **Derive** the control volume formulation of the continuity, momentum, and energy equations using Reynolds transport theorem and solving problems using those relations

15 **Derive** and use the Bernoulli equation (using the relation includes having knowledge about its limitations)

we will derive methods suitable for estimation of forces and system analysis

fluid flow finally ...

Roadmap - Integral Relations



Motivation

Fluid motion analysis:

differential approach (chapter 4):

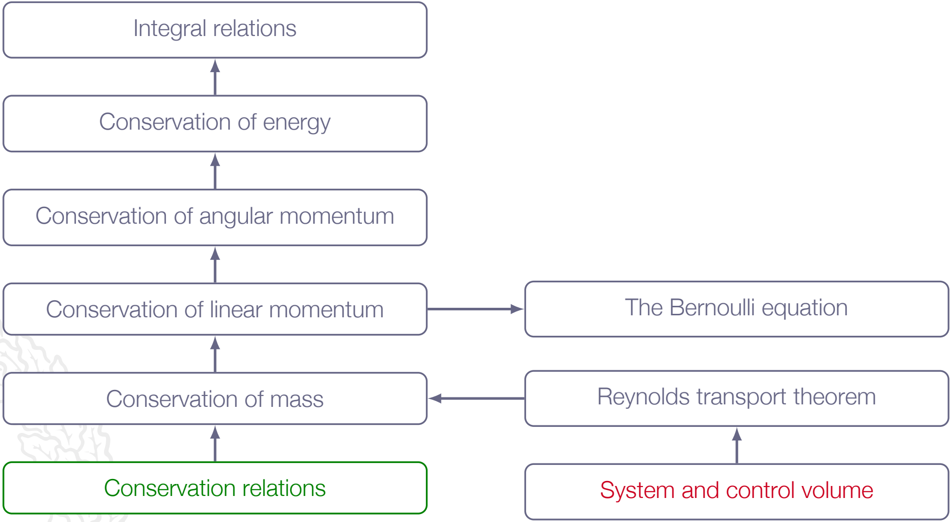
describe the detailed flow pattern at every point in the flow

control volume approach (chapter 3):

working with a finite region, balance in and out flow and determine gross flow effects (force, torque, energy exchange, ...)

gives useful engineering estimates

Roadmap - Integral Relations



System vs Control Volume

All laws of mechanics are written for a **system**:

- A system is an arbitrary quantity of **mass of fixed identity** m
- Mechanical relations describes interaction between the system and its surroundings

We need the corresponding relations expressed for a **control volume**:

- A region in space defined by the control volume surface
- Flow can pass through a control volume $\Rightarrow$ a system can pass through a control volume

Conservation Relations for a System

1. Conservation of mass
2. Conservation of linear momentum
3. Conservation of angular momentum
4. Conservation of energy

The above-listed relations includes thermodynamic properties and needs to be supplemented by a **state relation**

Remember: a thermodynamic property can be calculated from **any two other thermodynamics properties**

$$p = p(\rho, T), \; e = e(\rho, T)$$

Conservation of Mass for a System

$$m_{syst} = const$$

$$\frac{dm}{dt} = 0$$

Obvious in solid mechanics but needs attention in fluid mechanics

Conservation of Linear Momentum for a System

If the surroundings exert a net force $\mathbf{F}$ on the system, the mass in the system will begin to accelerate

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{V}}{dt} = \frac{d}{dt}(m\mathbf{V})$$

Conservation of Angular Momentum for a System

If the surroundings exert a net moment $\mathbf{M}$ about the center of mass of the system, there will be a rotation effect

$$\mathbf{M} = \frac{d\mathbf{H}}{dt}$$

where $\mathbf{H} = \Sigma(\mathbf{r} \times \mathbf{V})\delta m$ is the angular momentum of the system about its center of mass

Conservation of Energy for a System

First law of thermodynamics

$$\delta Q - \delta W = dE$$

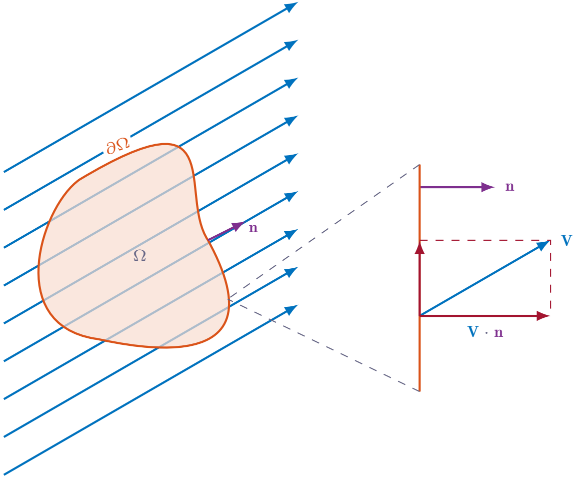
Second law of thermodynamics

$$dS \geq \frac{\delta Q}{T}$$

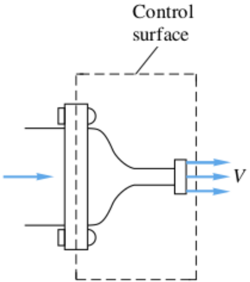
Control Volume - Volume and Mass Flow Rate

$$Q = \int_{\partial\Omega} (\mathbf{V} \cdot \mathbf{n}) dA$$

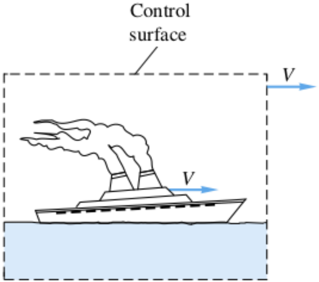
$$\dot{m} = \int_{\partial\Omega} \rho(\mathbf{V} \cdot \mathbf{n}) dA$$



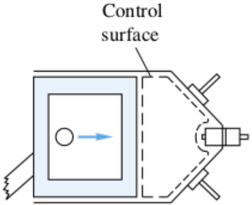
Control Volume - Classification



fixed

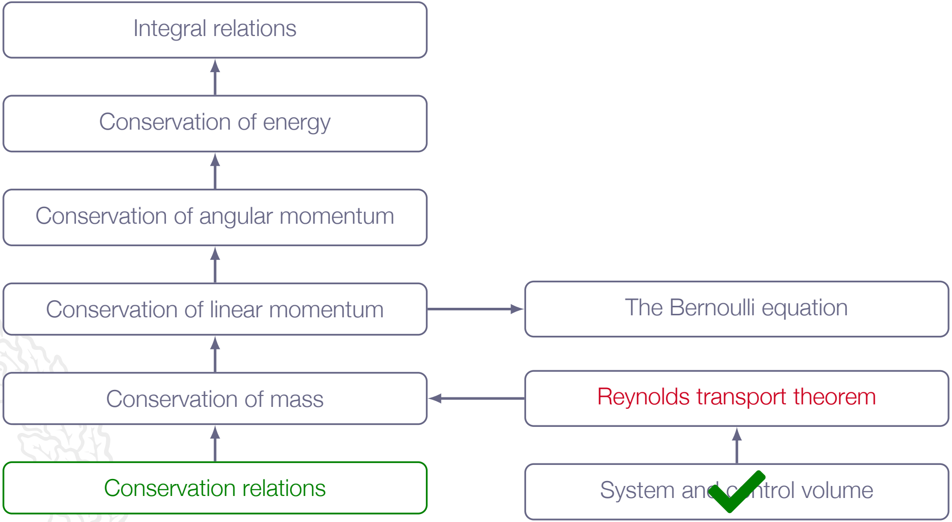


moving



deformable

Roadmap - Integral Relations



Reynolds Transport Theorem

Reynolds transport theorem

1. converts mathematical relations derived for a specific **system** to relations for a specific **region**
2. converts mathematical relations expressed in the **Lagrangian** frame of reference to relations expressed in the **Eulerian** frame of reference

Reynolds Transport Theorem

Let B be any **extensive** property of the fluid (energy, momentum, enthalpy, ...)

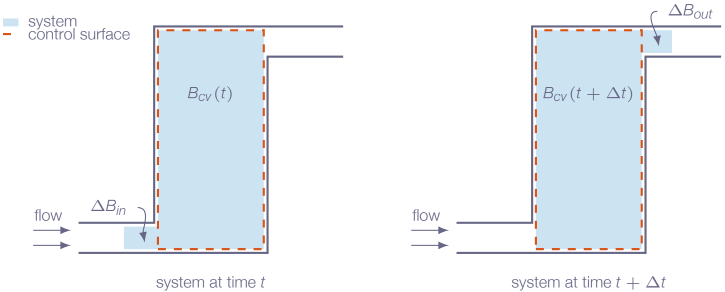
β is the corresponding **intensive** value (*the amount B per unit mass*)

The total amount of B in the control volume is

$$B_{CV} = \int_{CV} \beta dm = \int_{CV} \beta \rho d\mathcal{V}$$

where $\beta = \dfrac{dB}{dm}$

Reynolds Transport Theorem



$$B_{sys}(t) = B_{cv}(t) + \Delta B_{in}$$

$$B_{sys}(t+\Delta t) = B_{cv}(t+\Delta t) + \Delta B_{out}$$

Reynolds Transport Theorem

The rate of change of B for the system:

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \left[\frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} \right]$$

Apply relations from previous slide $\Rightarrow$

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0} \left[\frac{(B_{cv}(t + \Delta t) + \Delta B_{out}) - (B_{cv}(t) + \Delta B_{in})}{\Delta t} \right]$$

Reynolds Transport Theorem

Rewriting $\Rightarrow$

$$\frac{dB_{sys}}{dt} = \underbrace{\lim_{\Delta t \to 0}\left[\frac{B_{cv}(t+\Delta t) - B_{cv}(t)}{\Delta t}\right]}_{\frac{dB_{cv}}{dt}} + \underbrace{\underbrace{\lim_{\Delta t \to 0}\frac{\Delta B_{out}}{\Delta t}}_{\dot{B}_{out}} - \underbrace{\lim_{\Delta t \to 0}\frac{\Delta B_{in}}{\Delta t}}_{\dot{B}_{in}}}_{\dot{B}_{net}}$$

$$\frac{dB_{sys}}{dt} = \frac{dB_{cv}}{dt} + \dot{B}_{net}$$

Reynolds Transport Theorem

Rate of change of B within the control volume

$$\frac{dB_{cv}}{dt} = \frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right)$$

Net flux of B over the control volume surface

$$\dot{B}_{net} = \int_{CS} \beta\rho(\mathbf{V}\cdot\mathbf{n})dA$$

Reynolds Transport Theorem

$$\boxed{\underbrace{\frac{d}{dt}(B_{sys})}_{\textit{Lagrange}} = \underbrace{\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right) + \int_{CS}\beta\rho(\mathbf{V}\cdot\mathbf{n})dA}_{\textit{Euler}}}$$

Reynolds Transport Theorem

For a fixed control volume (the volume does not change in time)

$$\frac{d}{dt}\left(\int_{CV}\beta\rho d\mathcal{V}\right)=\int_{CV}\frac{\partial}{\partial t}(\beta\rho)d\mathcal{V}$$

Reynolds Transport Theorem

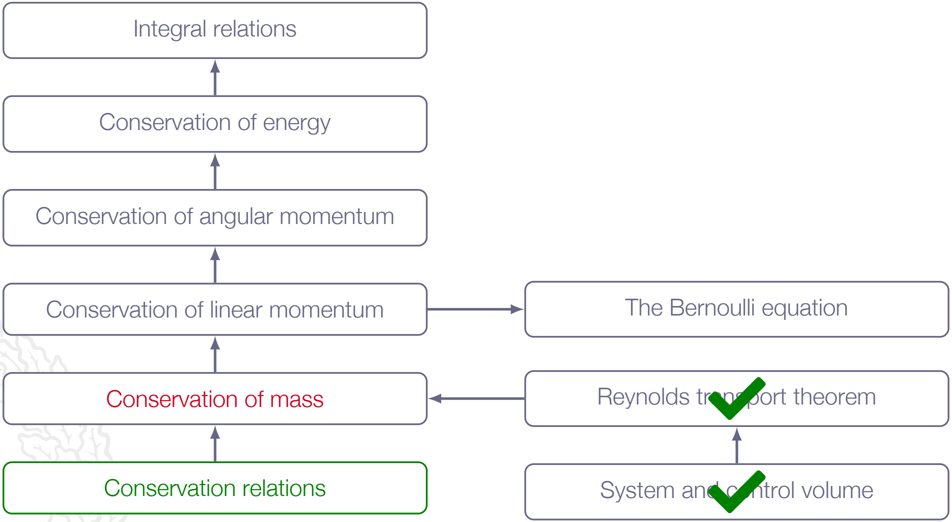
If the control volume moves with the constant velocity $\mathbf{V_s}$, the relative velocity of the fluid crossing the control volume surface $\mathbf{V_r}$ is

$$\mathbf{V_r} = \mathbf{V} - \mathbf{V_s}$$

and thus

$$\boxed{\frac{d}{dt}(B_{sys}) = \frac{d}{dt}\left(\int_{CV} \beta\rho d\mathcal{V}\right) + \int_{CS} \beta\rho(\mathbf{V_r} \cdot \mathbf{n})dA}$$

Roadmap - Integral Relations



Conservation of Mass

Reynolds transport theorem with $B = m$ and $\beta = dB/dm = dm/dm = 1$

$$\frac{d}{dt}(m_{sys}) = 0 = \frac{d}{dt}\left(\int_{CV} \rho d\mathcal{V}\right) + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

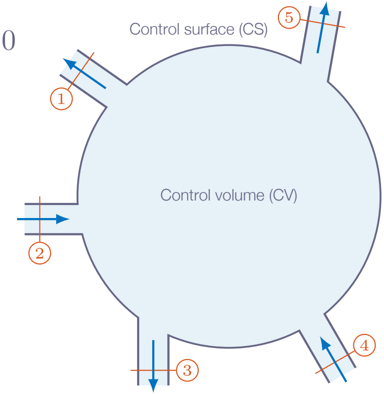
for a fixed control volume

$$\boxed{\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n})dA = 0}$$

Conservation of Mass

for a **fixed control volume** with a limited number of **one-dimensional** inlets and outlets

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$



Conservation of Mass

fixed control volume formulation

$$\boxed{\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n}) dA = 0}$$

Steady state $\Rightarrow \partial\rho/\partial t = 0$ and thus:

$$\int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n}) dA = 0$$

for a limited number of one-dimensional inlets and outlets:

$$\sum_i (\rho_i A_i V_i)_{out} = \sum_i (\rho_i A_i V_i)_{in}$$

Conservation of Mass

fixed control volume formulation

$$\boxed{\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho(\mathbf{V_r} \cdot \mathbf{n}) dA = 0}$$

Incompressible flow $\Rightarrow \rho = const \Rightarrow \partial\rho/\partial t = 0$ and thus:

$$\int_{CS} (\mathbf{V_r} \cdot \mathbf{n}) dA = 0$$

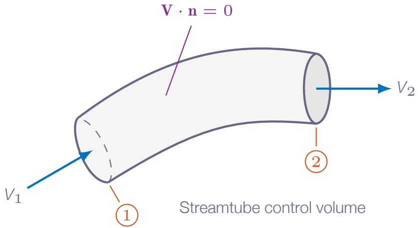
for a limited number of one-dimensional inlets and outlets:

$$\sum_i (A_i V_i)_{out} = \sum_i (A_i V_i)_{in}$$

Conservation of Mass - Example 1

Steady flow through a streamtube:

1. steady state $\Rightarrow$ no changes in time
2. streamtube $\Rightarrow$ only flow through the surfaces 1 and 2



$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = const$$

if the density is constant (incompressible flow)

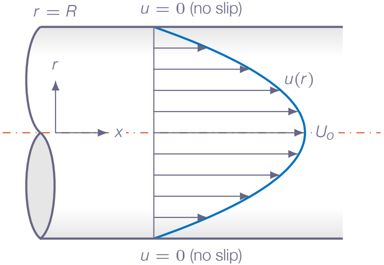
$$Q = A_1 V_1 = A_2 V_2 = const \Rightarrow V_2 = \frac{A_1}{A_2} V_1$$

Remember: a streamtube is constructed from a set of streamlines

Conservation of Mass - Example 2

Compute the average velocity for a steady **laminar** incompressible viscous flow through a circular tube with given axial velocity profile

$$u = U_o \left(1 - \left(\frac{r}{R}\right)^2\right)$$



Assumptions:

1. Laminar flow
2. Steady state $\Rightarrow$ no changes in time
3. Incompressible $\Rightarrow$ constant density

$$V_{av} = \frac{1}{A}\int u dA = \frac{1}{\pi R^2}\int_0^R U_o \left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r dr = \frac{2U_o}{R^2}\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr$$

Conservation of Mass - Example 2

$$V_{av} = \frac{2U_o}{R^2}\int_0^R \left(1-\left(\frac{r}{R}\right)^2\right) rdr = \frac{2U_o}{R^2}\left[\frac{r^2}{2}-\frac{r^4}{4R^2}\right]_0^R = \frac{U_o}{2}$$

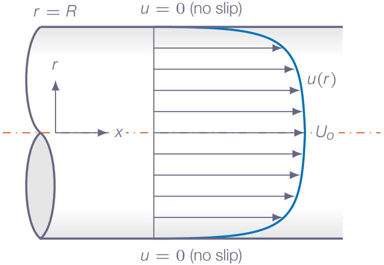
Thus, for **laminar** pipe flow

$$V_{av} = \frac{U_o}{2}$$

Conservation of Mass - Example 3

Compute the average velocity for a steady **turbulent** incompressible viscous flow through a circular tube with given axial velocity profile

$$u \approx U_o \left(1 - \frac{r}{R}\right)^m$$



Assumptions:

1. Turbulent flow: $1/5 \geq m \geq 1/9$
2. Steady state $\Rightarrow$ no changes in time
3. Incompressible $\Rightarrow$ constant density

$$V_{av} \approx \frac{1}{A} \int u dA = \frac{1}{\pi R^2} \int_0^R U_o \left(1 - \frac{r}{R}\right)^m 2\pi r dr = \frac{2U_o}{R^2} \int_0^R \left(1 - \frac{r}{R}\right)^m r dr$$

Conservation of Mass - Example 3

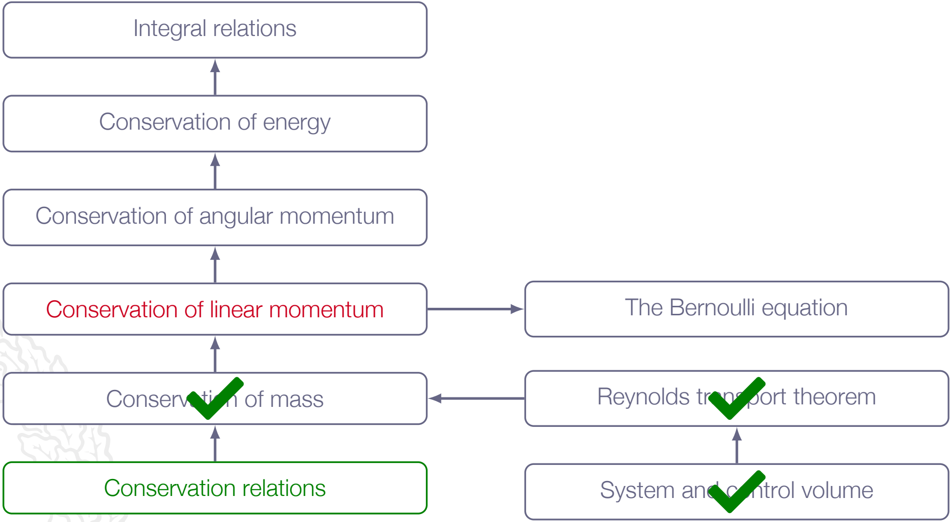
$$V_{av} \approx \frac{2U_o}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^m r dr = \frac{2U_o}{R^2}\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^m (mr+r+R)}{(m+1)(m+2)}\right]_0^R$$

Thus, for **turbulent** pipe flow

$$V_{av} \approx \frac{2U_o}{(m+1)(m+2)}$$

$m = 1/7 \Rightarrow V_{av} \approx 49U_o/60 \approx 0.82U_o$

Roadmap - Integral Relations



Conservation of Linear Momentum

Conservation of Linear Momentum

Reynolds transport theorem with $B = m\mathbf{V}$ and $\beta = dB/dm = d(m\mathbf{V})/dm = \mathbf{V}$

$$\boxed{\frac{d}{dt}(m\mathbf{V})_{sys} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V_r}\cdot\mathbf{n})dA}$$

1. $\mathbf{V}$ is the velocity relative to an inertial (non-accelerating) coordinate system
2. $\sum \mathbf{F}$ is the vector sum of all forces on the system (surface forces and body forces)
3. the relation is a vector relation (three components)

Conservation of Linear Momentum

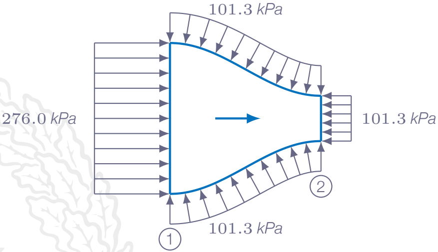
Forces on the system $\mathbf{F}$:

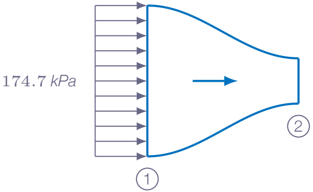
1. body forces (gravity, magnetic fields, coriolis forces)
2. forces transferred to the system via solid bodies that protrude through the control volume surface
3. forces due to pressure and viscous stresses of the surrounding fluid

Surface Pressure Force

$$\mathbf{F}_p = \int_{CS} p(-\mathbf{n})dA$$

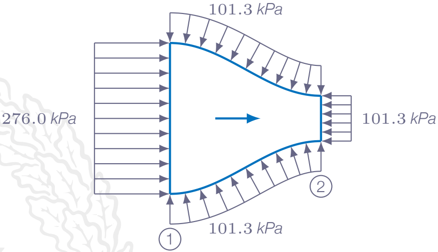
$$\mathbf{F}_p = \int_{CS} (p - p_{atm})(-\mathbf{n})dA = \int_{CS} p_{gage}(-\mathbf{n})dA$$

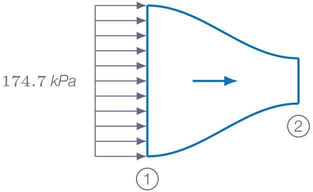




Surface Pressure Force

A free jet leaving a confined duct and exits into the ambient atmosphere will be at atmospheric pressure





Linear Momentum - Example

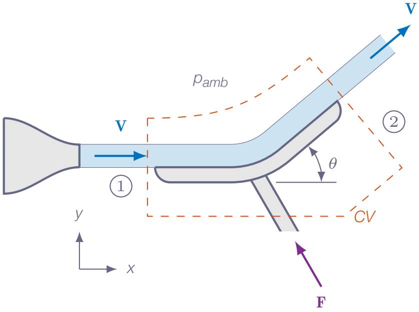
Deflection av a **steady-state** water jet without changing its velocity magnitude

1. steady-state
2. water $\Rightarrow$ incompressible
3. atmospheric pressure on all control volume surfaces
4. neglect friction

$$\mathbf{F} = \dot{m}_2\mathbf{V}_2 - \dot{m}_1\mathbf{V}_1$$

$$|\mathbf{V}_1| = |\mathbf{V}_2| = V$$

mass conservation: $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho A V$

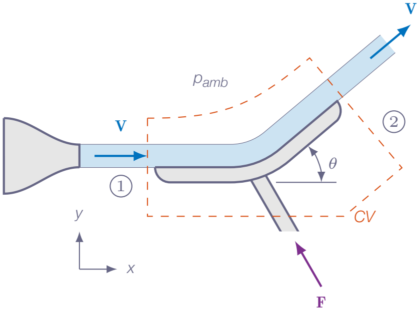


Linear Momentum - Example

$$F_x = \dot{m}V(\cos\theta - 1)$$

$$F_y = \dot{m}V\sin\theta$$

$$\mathbf{F} = \dot{m}V(\cos\theta - 1, \sin\theta, 0)$$



Momentum Flux Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the **momentum flux correction factor** ζ

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \zeta V_{av}\dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A}\int u dA$$

Momentum Flux Correction Factor

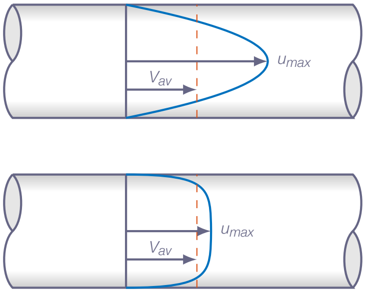
Laminar pipe flow:

Velocity profile:

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

From example 2 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{1}{2}U_{max}$$



Momentum Flux Correction Factor

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \int_0^R U_{max}^2\left(1 - \left(\frac{r}{R}\right)^2\right)\rho U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)2\pi r dr$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = 2\pi\rho U_{max}^2\int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right)^2 r dr$$

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{3}\pi R^2\rho U_{max}^2$$

Momentum Flux Correction Factor

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{3}\rho\pi R^2 U_{max}^2$$

the mass flow $\dot{m}$ can be obtained as:

$$\dot{m} = V_{av}\rho\pi R^2 = \frac{1}{2}U_{max}\rho\pi R^2$$

which gives

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{2U_{max}}{3}\dot{m}$$

from the definition of ζ

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\mathbf{n})dA = \zeta V_{av}\dot{m} = \left\{V_{av} = \frac{1}{2}U_{max}\right\} = \frac{\zeta U_{max}}{2}\dot{m}$$

Momentum Flux Correction Factor

comparing the two expressions, we have that

$$\frac{2U_{max}}{3}\dot{m} = \frac{\zeta U_{max}}{2}\dot{m}$$

and thus $\zeta = 4/3$ for **laminar incompressible** flow

Momentum Flux Correction Factor

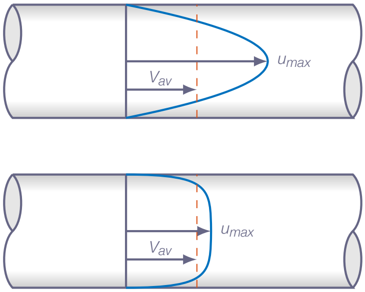
Turbulent pipe flow:

Velocity Profile:

$$u(r) \approx U_{max}\left(1 - \frac{r}{R}\right)^{m}, \; m \approx \frac{1}{7}$$

From example 3 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$



Momentum Flux Correction Factor

$$\rho 2\pi U_{max}^2 \int_0^R \left(1-\frac{r}{R}\right)^{2m} rdr = \zeta\rho\pi R^2 V_{av}^2 = \zeta\frac{4\rho\pi R^2 U_{max}^2}{(1+m)^2(2+m)^2} \Rightarrow$$

$$2\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^{2m}(2mr+r+R)}{2(1+2m)(1+m)}\right]_0^R = \zeta\frac{4R^2}{(1+m)^2(2+m)^2} \Rightarrow$$

$$\frac{R^2}{(1+2m)(1+m)} = \zeta\frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow \zeta = \frac{(1+m)^2(2+m)^2}{4(1+2m)(1+m)}$$

Momentum Flux Correction Factor

Laminar pipe flow:

$\zeta = 4/3$ should be used

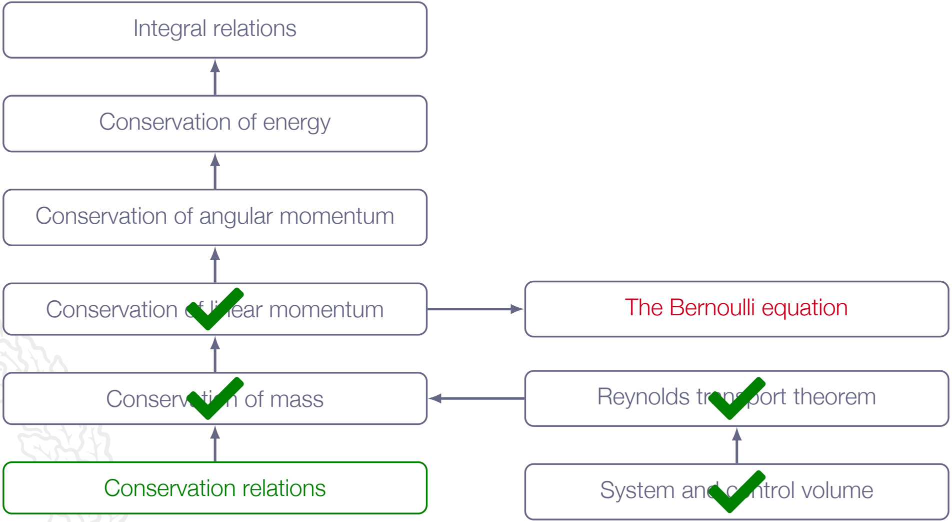
Turbulent pipe flow:

$$\zeta = \frac{(1+m)^2(2+m)^2}{4(1+2m)(1+m)}$$

m	1/5	1/6	1/7	1/8	1/9
ζ	1.037	1.027	1.020	1.016	1.013

$\zeta = 1.0$ is often a good approximation for turbulent flows

Roadmap - Integral Relations

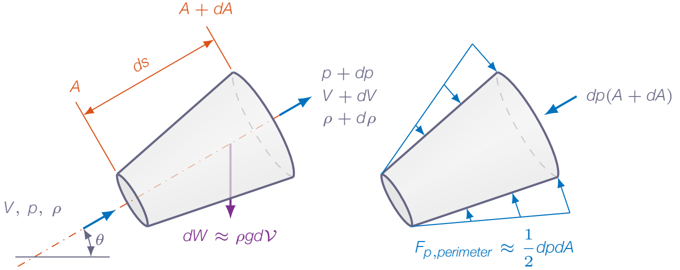


Daniel Bernoulli

The relation between pressure, velocity, and elevation in a frictionless flow

The Bernoulli Equation

Frictionless flow along a **streamline** (a streamtube with infinitesimal cross section area)



conservation of mass:

$$\frac{d}{dt}\left(\int_{CV} \rho d\mathcal{V}\right) + \dot{m}_{out} - \dot{m}_{in} = 0 \approx \frac{\partial \rho}{\partial t} d\mathcal{V} + d\dot{m}$$

where $d\mathcal{V} \approx Ads$

$$d\dot{m} \approx -\frac{\partial \rho}{\partial t} Ads$$

The Bernoulli Equation

linear momentum equation in the streamwise direction:

$$\sum dF_s = \frac{d}{dt}\left(\int_{CV} V\rho d\mathcal{V}\right) + (\dot{m}V)_{out} - (\dot{m}V)_{in} \approx \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

frictionless flow: only pressure and gravity forces

$$dF_{s,p} \approx \frac{1}{2}dpdA - (A + dA)dp \approx -Adp$$

$$dF_{s,grav} = -dW\sin\theta = -(g\rho A)ds\sin\theta = -g\rho Adz$$

$$\sum dF_s = -g\rho Adz - Adp = \frac{\partial}{\partial t}(\rho V)Ads + d(\dot{m}V)$$

The Bernoulli Equation

$$-g\rho Adz - Adp = \frac{\partial \rho}{\partial t}VAds + \frac{\partial V}{\partial t}\rho Ads + \underbrace{\dot{m}dV}_{=\rho AVdV} + Vd\dot{m}$$

the continuity equation gives

$$V\left[\frac{\partial \rho}{\partial t}Ads + d\dot{m}\right] \approx 0$$

and thus

$$\frac{\partial V}{\partial t}\rho Ads + Adp + \rho AVdV + g\rho Adz = 0$$

Now, divide by ρA

$$\frac{\partial V}{\partial t}ds + \frac{dp}{\rho} + VdV + gdz = 0$$

The Bernoulli Equation

Bernoulli's equation for unsteady **frictionless flow along a streamline** (the relation just derived) can be integrated between any two points along the streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0$$

The Bernoulli Equation

Steady ($\partial V / \partial t = 0$), **incompressible** (constant density) flow:

$$\boxed{p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = const}$$

The Bernoulli Equation

Note! the following restrictive assumptions have been made in the derivation

1. steady flow

   many flows can be treated as steady at least when doing control volume type of analysis

2. incompressible flow

   low velocity gas flow without significant changes in pressure, liquid flow

3. frictionless flow

   friction is in general important

4. flow along a single streamline

   different streamlines in general have different constants, we shall see later that under specific circumstances all streamlines have the same constant

One should be aware of these restrictions when using the Bernoulli relation

Relation to the Energy Equation

$$p_1 + \frac{1}{2}\rho V_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \rho g z_2 = const$$

Derived from the momentum equation

May be interpreted as a idealized energy equation (viscous dissipation not included) - provides a balance of

1. reversible pressure work
2. kinetic energy
3. potential energy

Stagnation, Static, and Dynamic Pressures

In many flows, elevation changes are negligible

$$p_1 + \frac{1}{2}\rho V_1^2 = p_2 + \frac{1}{2}\rho V_2^2 = p_o$$

Static pressure: p_1 and p_2

Dynamic pressure: $\frac{1}{2}\rho V_1^2$ and $\frac{1}{2}\rho V_2^2$

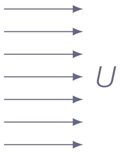
Stagnation (total) pressure: p_o

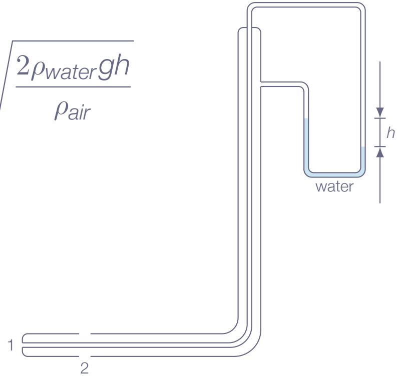
Pitot Static Tube

Pitot Static Tube

$$p_1 + \frac{1}{2}\rho_{air}U_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho_{air}U_2^2 + \rho g z_2$$

$$\left.\begin{aligned} U_1 &= 0. \\ U_2 &= U \\ z_1 &\approx z_2 \\ p_1 - p_2 &= \rho_{water} g h \end{aligned}\right\} \Rightarrow U = \sqrt{\frac{2\rho_{water} g h}{\rho_{air}}}$$





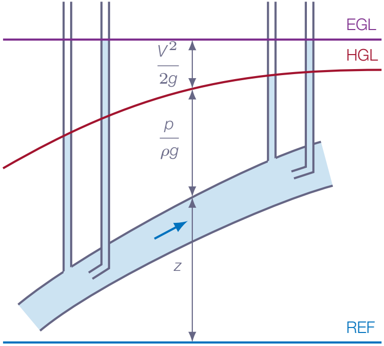
Hydraulic and Energy Grade Lines

EGL: $\dfrac{p}{\rho g} + \dfrac{V^2}{2g} + z$

EGL constant if:

1. the there are no friction losses
2. no heat is added or removed
3. no work is done

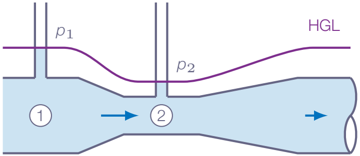
HGL: $\dfrac{p}{\rho g} + z =$ EGL - $\dfrac{V^2}{2g}$



Venturi Tube

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$z_1 = z_2$ gives



$$V_2^2 - V_1^2 = \frac{2\Delta p}{\rho}$$

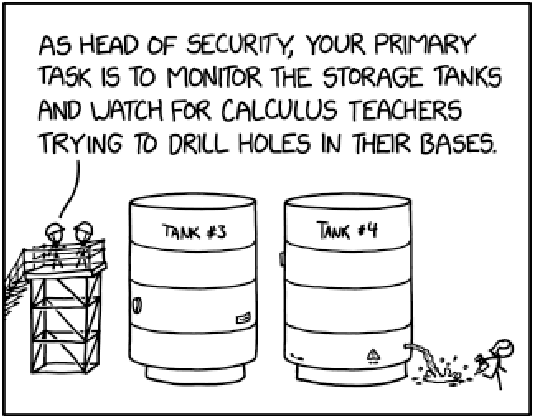
continuity:

$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2 = \frac{D_2^2}{D_1^2}V_2$$

inserted in the Bernoulli equation, this gives

$$V_2 = \left[\frac{2D_1^4\Delta p}{\rho(D_1^4 - D_2^4)}\right]^{1/2} \Rightarrow \dot{m} = \rho A_2V_2 = \frac{\pi D_1^2D_2^2}{4}\left[\frac{2\rho\Delta p}{D_1^4 - D_2^4}\right]^{1/2}$$

Tank Problem



Tank Problem - Solution 1

conservation of mass:

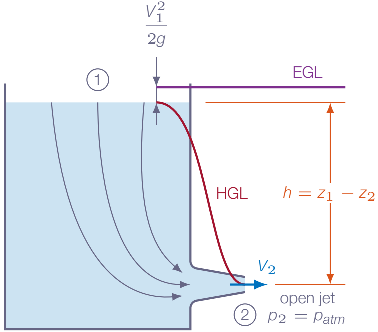
$$A_1V_1 = A_2V_2 \Rightarrow V_1 = \frac{A_2}{A_1}V_2$$

Bernoulli:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$p_1 = p_2 = p_{atm}$$

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh$$

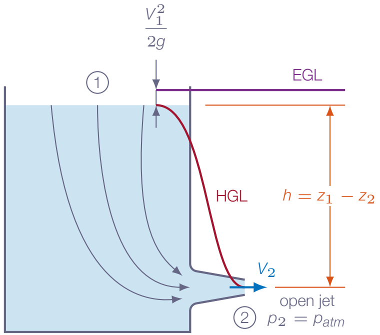


$$V_2 = \sqrt{\frac{2gh}{1 - \left(\frac{A_2}{A_1}\right)^2}}$$

$$A_2 \ll A_1 \Rightarrow V_2 \approx \sqrt{2gh}$$

Tank Problem - Solution 2

The outflow is very small in compared to the tank volume and thus the water surface hardly moves at all, *i.e.* $V_1 \approx 0$



Bernoulli:

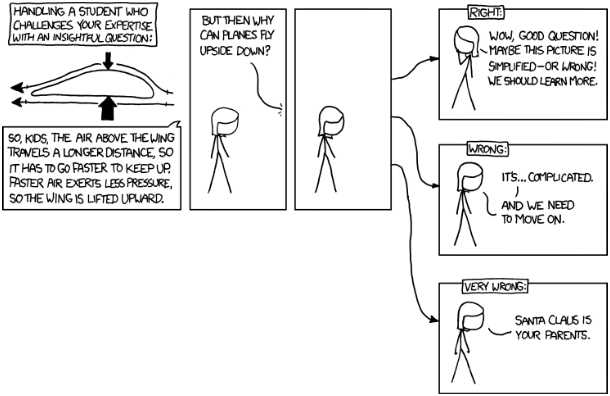
$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2$$

$$V_1 \approx 0, \; p_1 = p_2 = p_{atm}$$

$$V_2^2 = 2g(z_1 - z_2) = 2gh$$

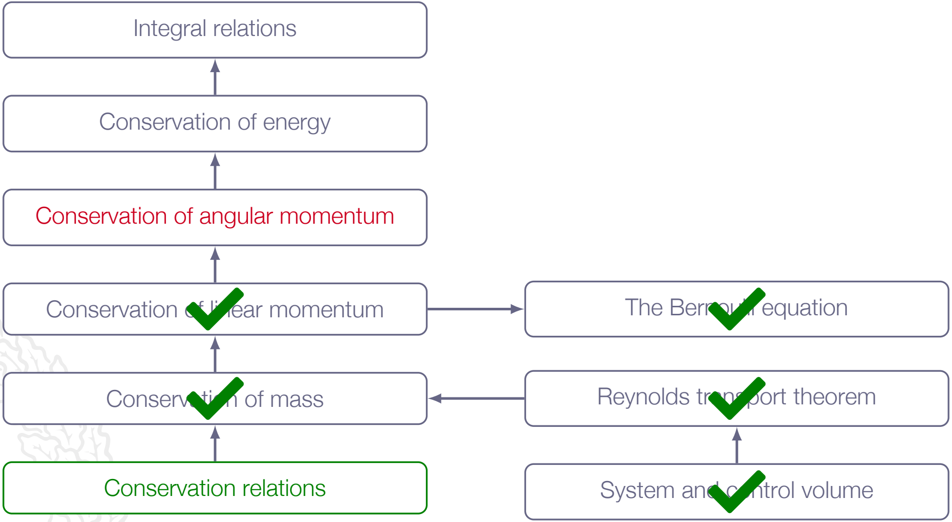
$$V_2 = \sqrt{2gh}$$

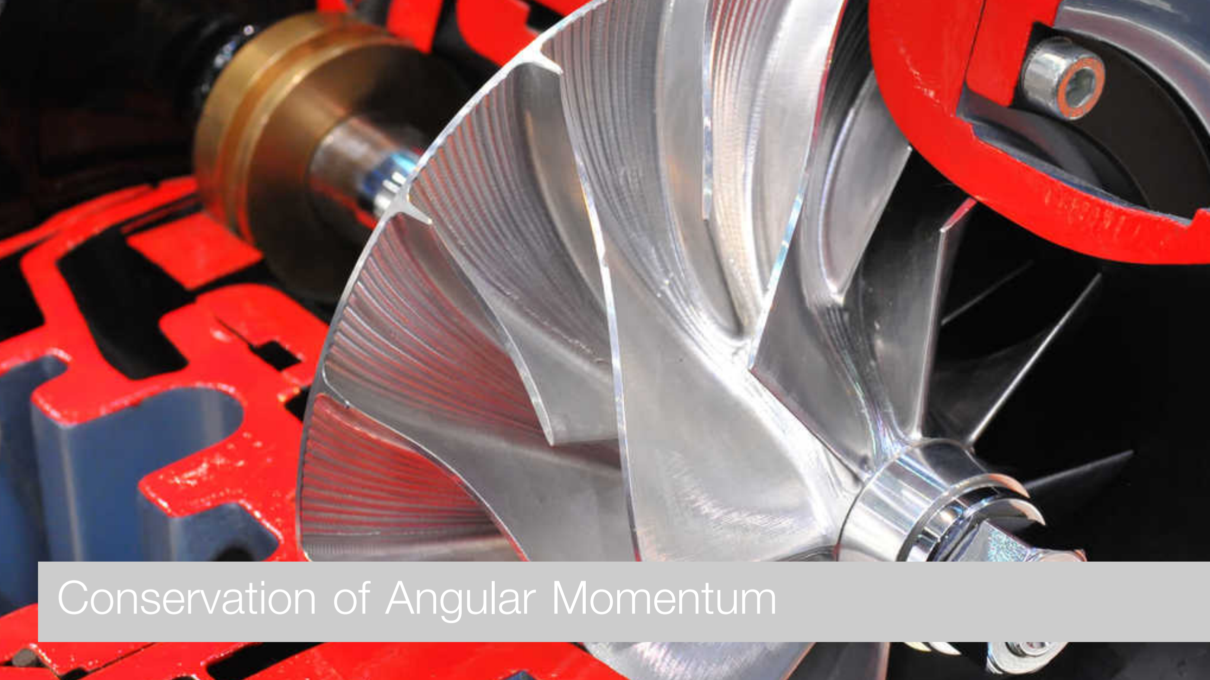
Airfoil



HANDLING A STUDENT WHO CHALLENGES YOUR EXPERTISE WITH AN INSIGHTFUL QUESTION:
SO, KIDS, THE AIR ABOVE THE WING TRAVELS A LONGER DISTANCE, SO IT HAS TO GO FASTER TO KEEP UP. FASTER AIR EXERTS LESS PRESSURE, SO THE WING IS LIFTED UPWARD.
BUT THEN WHY CAN PLANES FLY UPSIDE DOWN?
RIGHT:
WOW, GOOD QUESTION! MAYBE THIS PICTURE IS SIMPLIFIED—OR WRONG! WE SHOULD LEARN MORE.
WRONG:
IT'S... COMPLICATED.
AND WE NEED TO MOVE ON.
VERY WRONG:
SANTA CLAUS IS YOUR PARENTS.

Roadmap - Integral Relations





Conservation of Angular Momentum

Angular Momentum

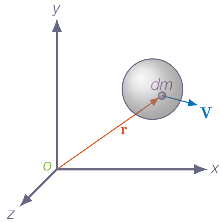
Angular momentum about a point o

$$\mathbf{H}_o = \int_{syst} (\mathbf{r} \times \mathbf{V}) dm = B$$

where $\mathbf{r}$ is the position vector from o to the element mass dm and $\mathbf{V}$ is the velocity of that element

The amount of angular momentum per unit mass

$$\beta = \frac{d\mathbf{H}_o}{dm} = \mathbf{r} \times \mathbf{V}$$



Conservation of Angular Momentum

Reynold's transport theorem:

$$\left.\frac{d\mathbf{H}_o}{dt}\right|_{syst} = \frac{d}{dt}\left[\int_{CV}(\mathbf{r}\times\mathbf{V})\rho d\mathcal{V}\right] + \int_{CS}(\mathbf{r}\times\mathbf{V})\rho(\mathbf{V}_r\cdot\mathbf{n})dA$$

for inertial coordinate systems:

$$\frac{d\mathbf{H}_o}{dt} = \sum\mathbf{M}_o = \sum(\mathbf{r}\times\mathbf{F})_o$$

Conservation of Angular Momentum

Non-deformable inertial control volume:

$$\boxed{\sum \mathbf{M}_o = \frac{d}{dt}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho d\mathcal{V}\right] + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) dA}$$

for a limited number of **one-dimensional** inlets and outlets

$$\int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) dA = \sum (\mathbf{r} \times \mathbf{V})_{out}\dot{m}_{out} - \sum (\mathbf{r} \times \mathbf{V})_{in}\dot{m}_{in}$$

Angular Momentum Example - Garden Sprinkler

Assumptions:

1. steady-state flow
2. incompressible flow (water)

Angular Momentum Example - Garden Sprinkler

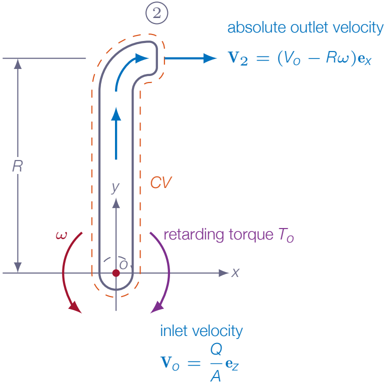
$$\sum \mathbf{M}_o = \underbrace{\frac{d}{dt}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho d\mathcal{V}\right]}_{=0 \ (steady \ state)} + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V_r} \cdot \mathbf{n})dA$$

$$\mathbf{V}_2 = (V_o - R\omega, 0, 0)$$

$$\mathbf{V}_o = (0, 0, V_o)$$

$$\mathbf{r}_2 = (0, R, 0)$$

$$\mathbf{r}_o = (0, 0, 0)$$



Angular Momentum Example - Garden Sprinkler

$$\sum \mathbf{M}_o = \int_{CS} (\mathbf{r} \times \mathbf{V}) \rho (\mathbf{V_r} \cdot \mathbf{n}) dA$$

inlet:

$$(\mathbf{r}_o \times \mathbf{V}_o) = (0,0,0) \times (0,0,V_o) = (0,0,0)$$

$$(\mathbf{V}_{o,r} \cdot \mathbf{n}_o) = (0,0,V_o) \cdot (0,0,-1) = -V_o$$

$$(\mathbf{r}_o \times \mathbf{V}_o) \rho (\mathbf{V}_o \cdot \mathbf{n}_o) A_o = -(0,0,0) \rho V_o A_o = (0,0,0)$$

outlet:

$$(\mathbf{r}_2 \times \mathbf{V}_2) = (0,R,0) \times (V_o - R\omega, 0, 0) = (0,0,R^2\omega - RV_o)$$

$$(\mathbf{V}_{2,r} \cdot \mathbf{n}_2) = (V_o,0,0) \cdot (1,0,0) = V_o$$

$$(\mathbf{r}_2 \times \mathbf{V}_2) \rho (\mathbf{V}_2 \cdot \mathbf{n}_2) A_2 = (0,0,R^2\omega - RV_o) \rho V_o A_2 = \rho Q (0,0,R^2\omega - RV_o)$$

Angular Momentum Example - Garden Sprinkler

$$\sum \mathbf{M}_o = (0, 0, -T_o) = \rho Q(0, 0, R^2\omega - RV_o)$$

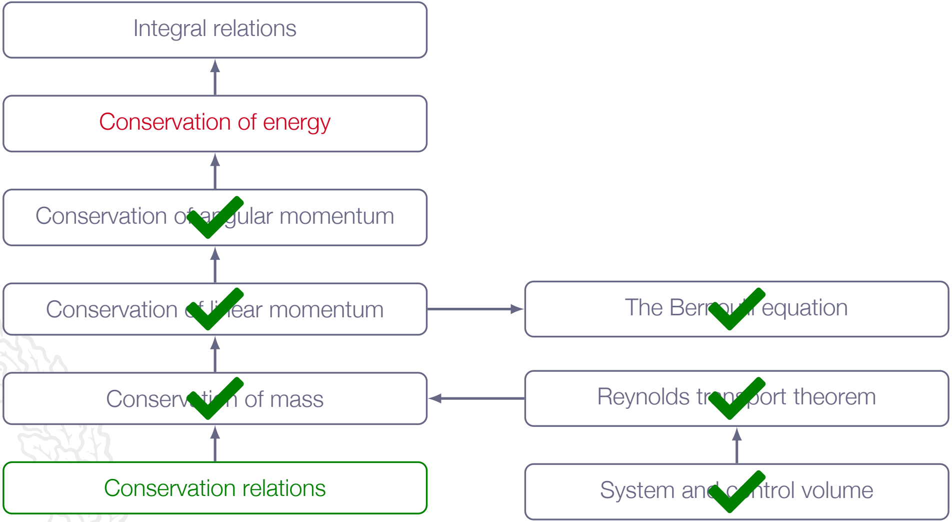
$$-T_o = \rho Q R^2 \left(\omega - \frac{V_o}{R}\right) \Rightarrow \omega = \frac{V_o}{R} - \frac{T_o}{\rho Q R^2}$$

Note:

with a negligible retarding torque, *i.e.* $T_o \approx 0.$, we get $\omega \approx \omega|_{T_o=0.} = \frac{V_o}{R}$ $[rad/s]$

the torque required to hold the sprinkler arm still is $T_o|_{\omega=0.} = \rho Q V_o R$ $[Nm]$

Roadmap - Integral Relations



The Energy Equation

Reynold's transport theorem applied the the **first law of thermodynamics** ($B = E$, $\beta = dE/dm = e$)

$$\frac{dQ_{sys}}{dt} - \frac{dW_{sys}}{dt} = \frac{dE_{sys}}{dt} = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} e\rho(\mathbf{V}\cdot\mathbf{n})dA$$

Recall:

positive Q_{sys}: heat added **to** the system

positive W_{sys}: work done **by** the system on its surroundings

The Energy Equation - Energy per Unit Mass

$$e = e_{internal} + e_{kinetic} + e_{potential} + e_{other}$$

e_{other} could be related to, for example, chemical reactions, nuclear reactions, or magnetic fields and will not be considered here

$$e = \hat{u} + \frac{1}{2}V^2 + gz$$

The Energy Equation - Work

The work term $\dot{W}$ can be divided into shaft work, pressure work, and work related to viscous forces

$$\dot{W} = \dot{W}_s + \dot{W}_p + \dot{W}_\nu$$

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

$$\dot{W}_\nu = -\int_{CS} \tau \cdot \mathbf{V} dA$$

The Energy Equation - Pressure Work

$$\dot{W}_p = \int_{CS} p(\mathbf{V} \cdot \mathbf{n})dA$$

The rate of work done by **pressure forces** on the **control volume surfaces**

internal forces will always have an opposite force leading to cancelation

The Energy Equation - Viscous Work

$$\dot{W}_\nu = -\int_{CS} \tau \cdot \mathbf{V} dA$$

The rate of work related to **viscous stresses** on the **control volume surfaces**

important or not depending on flow situation

The Energy Equation - Control Volume Boundaries

Solid walls:

no-slip $\Rightarrow \dot{W}_\nu = 0$

Machine surfaces:

viscous work included implicitly in shaft work

Inlets/outlets:

flow aligned with surface normal (usually) and normal viscous stress components are in most cases very small

Streamlines:

viscous stresses may be significant *depending on streamline location*

The Energy Equation

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} \rho e(\mathbf{V}\cdot\mathbf{n})dA + \int_{CS} p(\mathbf{V}\cdot\mathbf{n})dA$$

collecting surface integrals gives

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS} \left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

The Energy Equation

$$\boxed{\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left(\int_{CV} e\rho d\mathcal{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})dA}$$

or

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \frac{d}{dt}\left[\int_{CV}\left(\hat{u} + \frac{1}{2}V^2 + gz\right)\rho d\mathcal{V}\right] + \int_{CS}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\rho(\mathbf{V}\cdot\mathbf{n})dA$$

where $\hat{h}$ is the enthalpy defined as $\hat{h} = \hat{u} + p/\rho$

The Energy Equation

Steady state:

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = \int_{CS} \left(\hat{h} + \frac{1}{2}V^2 + gz\right) \rho(\mathbf{V} \cdot \mathbf{n}) dA$$

Special case: one inlet and one outlet (both one-dimensional)

$$\dot{Q} - \dot{W}_s - \dot{W}_\nu = -\dot{m}_1 \left(\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1\right) + \dot{m}_2 \left(\hat{h}_2 + \frac{1}{2}V_2^2 + gz_2\right)$$

continuity $\Rightarrow \dot{m}_1 = \dot{m}_2 = \dot{m}$, divide by $\dot{m}$ gives

$$\boxed{\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu}$$

The Energy Equation

$$\hat{h}_1 + \frac{1}{2}V_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}V_2^2 + gz_2 - q + w_s + w_\nu$$

all terms has the dimension $[m^2/s^2]$, divide by g $[m/s^2]$ to get dimension $[m]$

$$\frac{p_1}{\rho g} + \frac{\hat{u}_1}{g} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{\hat{u}_2}{g} + \frac{V_2^2}{2g} + z_2 - \frac{q}{g} + h_s + h_\nu$$

$p/(\rho g)$: pressure head

$V^2/2g$: velocity head

The Energy Equation

1. steady-state flow
2. incompressible (low speed)
3. pipe/duct that may or may not include turbines and pumps
4. solid walls $\Rightarrow h_\nu = 0$

$$\underbrace{\left(\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1\right)}_{h_{o_1}}=\underbrace{\left(\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2\right)}_{h_{o_2}}+\frac{\hat{u}_2-\hat{u}_1-q}{g}$$

where h_o is *available head* or *total head*

The Energy Equation

friction head losses h_f (always positive)

pump head input h_p

turbine head extraction h_t

$$\boxed{\left(\frac{p_1}{\rho g}+\frac{V_1^2}{2g}+z_1\right)_{in}=\left(\frac{p_2}{\rho g}+\frac{V_2^2}{2g}+z_2\right)_{out}+h_f-h_p+h_t}$$

Kinetic Energy Correction Factor

One-dimensional flow through inlets and outlets is of course not true in reality

Introducing the correction factor α

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{\alpha {V_{av}}^2}{2}\dot{m}$$

where (for incompressible flow)

$$V_{av} = \frac{1}{A}\int u dA$$

Kinetic Energy Correction Factor

$$\left(\frac{p_1}{\rho g} + \frac{\alpha_1 V_1^2}{2g} + z_1\right)_{in} = \left(\frac{p_2}{\rho g} + \frac{\alpha_2 V_2^2}{2g} + z_2\right)_{out} + h_f - h_p + h_t$$

Kinetic Energy Correction Factor

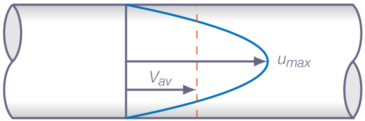
Laminar pipe flow:

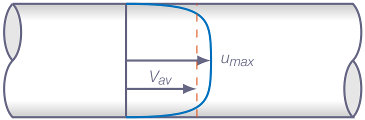
Velocity profile:

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

From example 2 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{1}{2}U_{max}$$





Kinetic Energy Correction Factor

$$u(r) = U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \int_0^R \frac{1}{2}U_{max}^2\left(1 - \left(\frac{r}{R}\right)^2\right)^2 \rho U_{max}\left(1 - \left(\frac{r}{R}\right)^2\right) 2\pi r dr$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \rho\pi U_{max}^3 \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right)^3 r dr$$

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{8}\rho\pi R^2 U_{max}^3$$

Kinetic Energy Correction Factor

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{1}{8}\rho\pi R^2 U_{max}^3$$

the mass flow $\dot{m}$ can be obtained as:

$$\dot{m} = V_{av}\rho\pi R^2 = \frac{1}{2}U_{max}\rho\pi R^2$$

which gives

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{U_{max}^2}{4}\dot{m}$$

from the definition of α

$$\int \frac{1}{2}V^2\rho(\mathbf{V}\cdot\mathbf{n})dA = \frac{\alpha V_{av}^2}{2}\dot{m} = \left\{V_{av} = \frac{1}{2}U_{max}\right\} = \frac{\alpha U_{max}^2}{8}\dot{m}$$

Kinetic Energy Correction Factor

comparing the two expressions, we have that

$$\frac{U_{max}^2}{4}\dot{m} = \frac{\alpha U_{max}^2}{8}\dot{m}$$

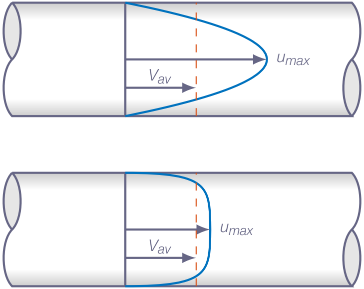
and thus $\alpha = 2$ for laminar incompressible flow

Kinetic Energy Correction Factor

Turbulent pipe flow:

Velocity Profile:

$$u(r) \approx U_{max}\left(1-\frac{r}{R}\right)^m,\ m \approx \frac{1}{7}$$



From example 3 in the conservation-of-mass section we have that the average velocity for a laminar pipe flow is obtained as:

$$V_{av} = \frac{2U_{max}}{(1+m)(2+m)}$$

Kinetic Energy Correction Factor

$$\rho\pi U_{max}^3 \int_0^R \left(1-\frac{r}{R}\right)^{3m} r dr = \alpha \frac{1}{2}\rho\pi R^2 V_{av}^3 = \alpha \frac{4\rho\pi R^2 U_{max}^3}{(1+m)^3(2+m)^3} \Rightarrow$$

$$\left[\frac{(r-R)\left(1-\frac{r}{R}\right)^{3m}(3mr+r+R)}{(1+3m)(2+3m)}\right]_0^R = \alpha \frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow$$

$$\frac{R^2}{(1+3m)(2+3m)} = \alpha \frac{4R^2}{(1+m)^3(2+m)^3} \Rightarrow \alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

Kinetic Energy Correction Factor

Laminar pipe flow:

$\alpha = 2.0$ should be used

Turbulent pipe flow:

$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)}$$

m	1/5	1/6	1/7	1/8	1/9
α	1.106	1.077	1.058	1.046	1.037

$\alpha = 1.0$ is often a good approximation for turbulent pipe flows

The Energy Equation - Pump Example

Task:

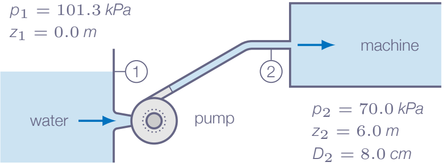
Calculate pump power if $\eta = 0.8$

Given:

1. Geometry and pressures from figure
2. The pump delivers water at a flow rate $Q = 0.04\ m^3/s$
3. Friction losses between 1 and 2 are given by $h_f = KV_2^2/(2g)$ where $K \approx 7.5$
4. $\alpha \approx 1.07$

Assumptions:

1. steady-state flow
2. negligible viscous work
3. large reservoir ($V_1 \approx 0$)



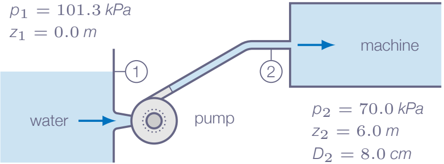
The Energy Equation - Pump Example

$$V_2 = \frac{Q}{A_2} = 7.96\ m/s$$

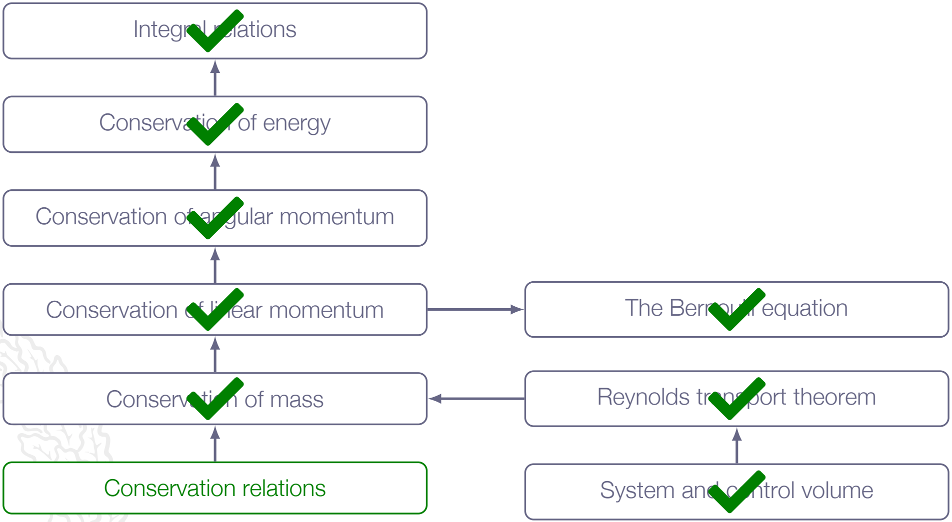
$$\left(\frac{p_1}{\rho g} + \frac{\alpha V_1^2}{2g} + z_1\right)_{in} = \left(\frac{p_2}{\rho g} + \frac{\alpha V_2^2}{2g} + z_2\right)_{out} + h_f - h_p + h_t$$

$$h_p = \frac{p_2 - p_1}{\rho g} + (z_2 - z_1) + (\alpha_2 + K)\frac{V_2^2}{2g} = 30.5\ m$$

$$P_{pump} = \frac{\rho g Q h_p}{\eta} = 14960\ W\ (or\ 20\ hp)$$



Roadmap - Integral Relations



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8. can inlets/outlets be assumed to be **one-dimensional**

Integral Relations