Compressible Flow - TME085

Chapter 5

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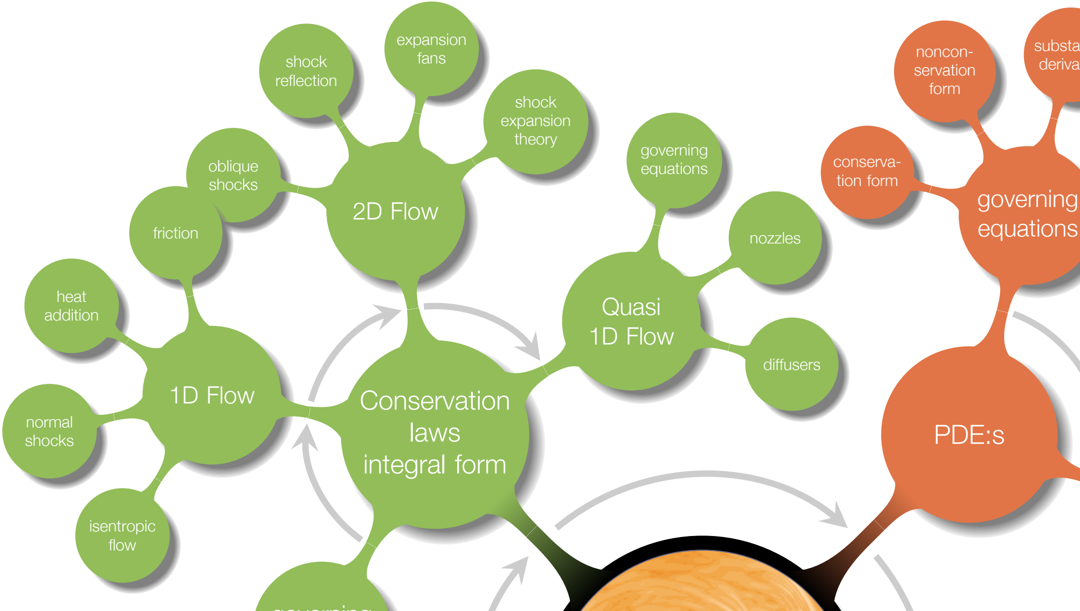
Chapter 5 - Quasi-One-Dimensional Flow

Overview



shock reflection
expansion fans
shock expansion theory
oblique shocks
2D Flow
friction
governing equations
nozzles
Quasi 1D Flow
diffusers
heat addition
1D Flow
normal shocks
isentropic flow
Conservation laws integral form
noncon-servation form
conserva-tion form
governing equations
PDE:s

Learning Outcomes

4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on

6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases

7 **Explain** why entropy is important for flow discontinuities

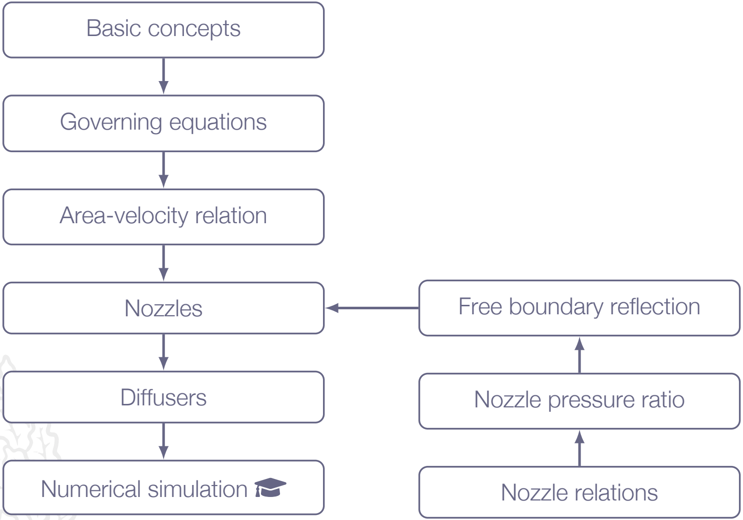
8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics

  a 1D isentropic flow*
  b normal shocks*
  i detached blunt body shocks, nozzle flows

9 **Solve** engineering problems involving the above-mentioned phenomena (8a-8k)

what does quasi-1D mean? either the flow is 1D or not, or?

Roadmap - Quasi-One-Dimensional Flow

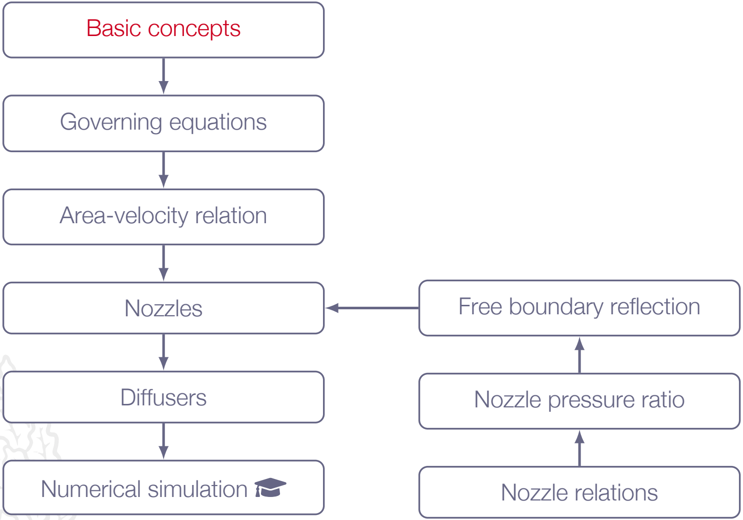


Motivation

By extending the one-dimensional theory to quasi-one-dimensional, we can study important applications such as nozzles and diffusers

Even though the flow in nozzles and diffusers are in essence three dimensional we will be able to establish important relations using the quasi-one-dimensional approach

Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Flow

Chapter 3

overall assumption

one-dimensional flow
steady state
constant cross-section area

applications

normal shock
1D flow with heat addition
1D flow with friction

Chapter 4

overall assumption

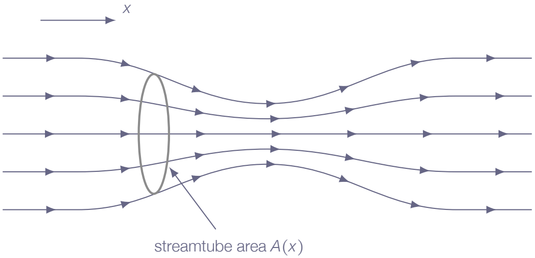
two-dimensional flow
steady state
uniform freestream

applications

oblique shocks
expansion fans
shock-expansion theory

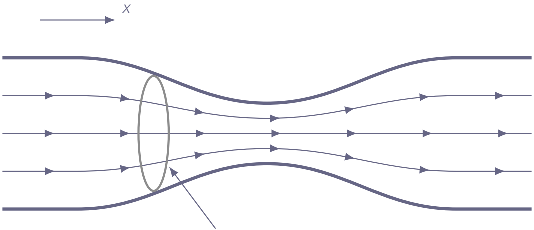
Quasi-One-Dimensional Flow

Extension of one-dimensional flow to allow **variations in streamtube area** (steady-state flow assumption still applied)



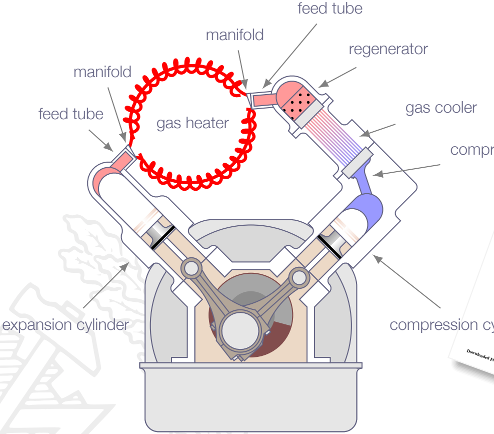
Quasi-One-Dimensional Flow

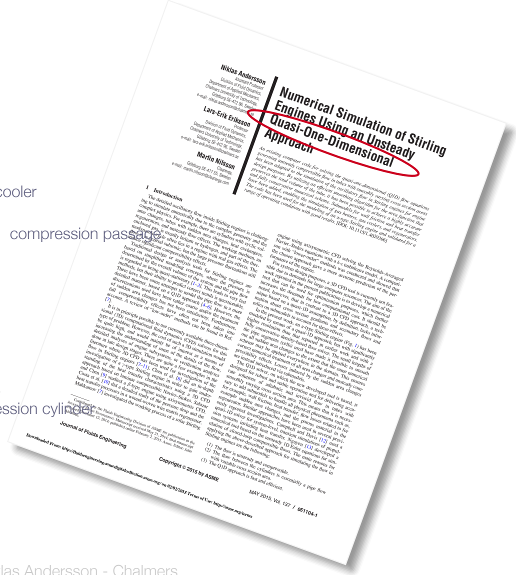
Example: tube with variable cross-section area



Quasi-One-Dimensional Flow - Nozzle Flow

Quasi-One-Dimensional Flow - Stirling Engine




Numerical Simulation of Stirling Engines Using an Unsteady Quasi-One-Dimensional Approach

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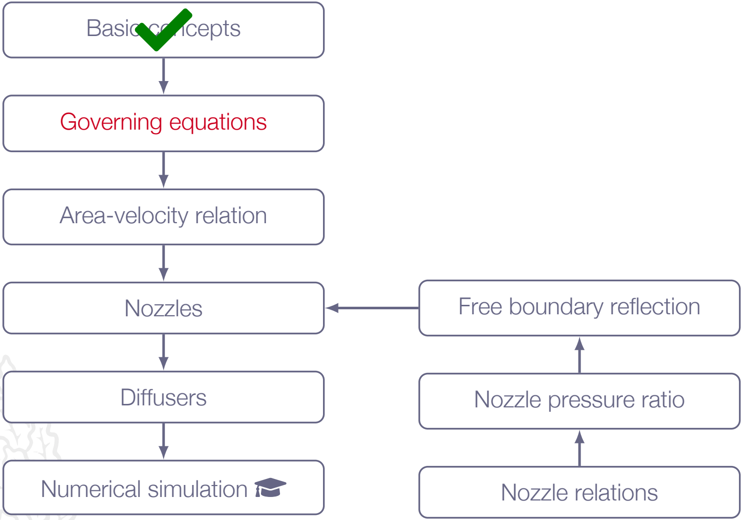
An existing computer code for solving the quasi-one-dimensional (Q1D) flow equations governing unsteady compressible flow in tubes with smoothly varying cross section areas has been adapted to the simulation of the oscillatory flow in Stirling engines for engine design purposes. By utilizing an efficient smoothing algorithm for the area function that preserves the total volume of the tube, it has been possible to achieve a highly accurate and fully conservative numerical scheme. Submodels for wall friction, gas coolers, heat transfer have been added, enabling the simulation of gas heaters, gas coolers, and regenerators. The code has been used for the modeling of an α-type Stirling engine and validated for a range of operating conditions with good results. [DOI: 10.1115/1.4029396]

1 Introduction

Roadmap - Quasi-One-Dimensional Flow

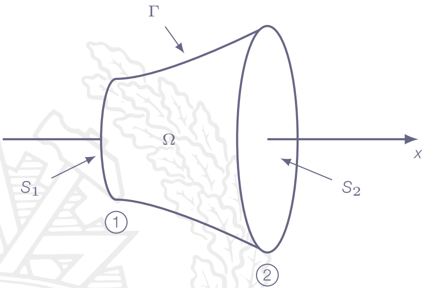


Chapter 5.2
Governing Equations

Governing Equations

Introduce **cross-section-averaged flow quantities** $\Rightarrow$
all quantities depend on x only

$$A = A(x),\ \rho = \rho(x),\ u = u(x),\ p = p(x),\ \ldots$$



Ω	control volume
S_1	left boundary (area A_1)
S_2	right boundary (area A_2)
Γ	perimeter boundary

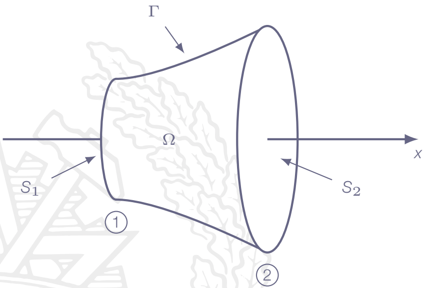
$$\partial\Omega = S_1 \cup \Gamma \cup S_2$$

Governing Equations - Assumptions

1. Inviscid flow (no boundary layers)

2. Steady-state flow (no unsteady effects)

3. No flow through Γ (control volume aligned with streamlines)



Governing Equations - Conservation of Mass

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho d\mathscr{V}}_{=0} + \underbrace{\oint\!\!\!\oint\limits_{\partial\Omega} \rho \mathbf{v}\cdot\mathbf{n}dS}_{-\rho_1 u_1 A_1 + \rho_2 u_2 A_2} = 0$$

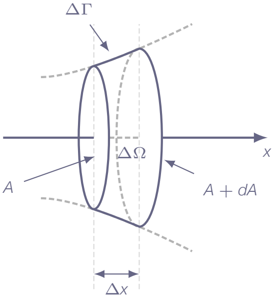
$$\boxed{\rho_1 u_1 A_1 = \rho_2 u_2 A_2}$$

Governing Equations - Conservation of Momentum

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega}\rho\mathbf{v}d\mathscr{V}}_{=0}+\oiint\limits_{\partial\Omega}\left[\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}+p\mathbf{n}\right]dS=0$$

$$\oiint\limits_{\partial\Omega}\rho(\mathbf{v}\cdot\mathbf{n})\mathbf{v}dS=-\rho_1u_1^2A_1+\rho_2u_2^2A_2$$

$$\oiint\limits_{\partial\Omega}p\mathbf{n}dS=-p_1A_1+p_2A_2-\int_{A_1}^{A_2}pdA$$



$$\boxed{(\rho_1u_1^2+p_1)A_1+\int_{A_1}^{A_2}pdA=(\rho_2u_2^2+p_2)A_2}$$

Governing Equations - Conservation of Energy

$$\underbrace{\frac{d}{dt}\iiint\limits_{\Omega} \rho e_o d\mathscr{V}}_{=0} + \oiint\limits_{\partial\Omega} [\rho h_o(\mathbf{v}\cdot\mathbf{n})]\, dS = 0$$

which gives

$$\rho_1 u_1 A_1 h_{o_1} = \rho_2 u_2 A_2 h_{o_2}$$

from continuity we have that $\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \Rightarrow$

$$\boxed{h_{o_1} = h_{o_2}}$$

Governing Equations - Summary

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

$$(\rho_1 u_1^2 + p_1) A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2) A_2$$

$$h_{o_1} = h_{o_2}$$

Governing Equations - Differential Form

Continuity equation:

$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$ or $\rho u A = c$

where c is a constant $\Rightarrow$

$$\boxed{d(\rho u A) = 0}$$

Governing Equations - Differential Form

Momentum equation:

$$(\rho_1 u_1^2 + p_1)A_1 + \int_{A_1}^{A_2} p dA = (\rho_2 u_2^2 + p_2)A_2 \Rightarrow$$

$$d\left[(\rho u^2 + p)A\right] = pdA \Rightarrow$$

$$d(\rho u^2 A) + d(pA) = pdA \Rightarrow$$

$$u\underbrace{d(\rho uA)}_{=0} + \rho uAdu + Adp + pdA = pdA \Rightarrow$$

$$\rho uAdu + Adp = 0 \Rightarrow$$

$$\boxed{dp = -\rho udu}$$ (Euler's equation)

Governing Equations - Differential Form

Energy equation:

$$h_{o_1} = h_{o_2} \Rightarrow dh_o = 0$$

$$h_o = h + \frac{1}{2}u^2 \Rightarrow$$

$$\boxed{dh + udu = 0}$$

Governing Equations - Differential Form

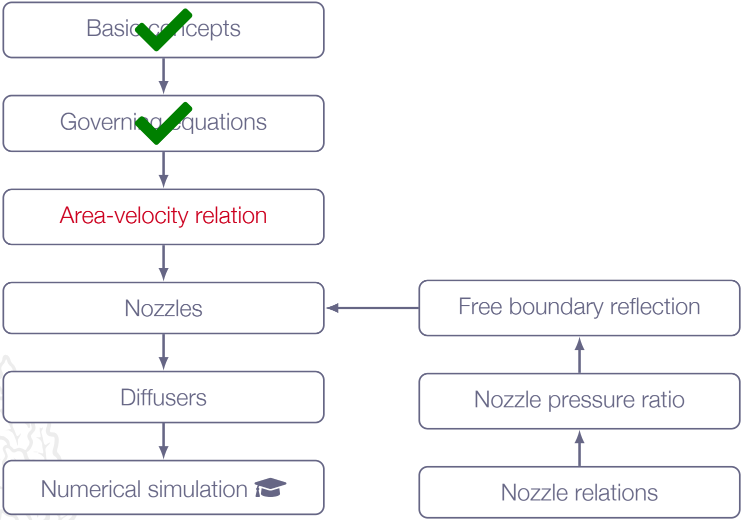
Summary (valid for all gases):

$$\boxed{\begin{aligned} d(\rho u A) &= 0 \\ dp &= -\rho u du \\ dh + u du &= 0 \end{aligned}}$$

Assumptions:

1. quasi-one-dimensional flow
2. inviscid flow
3. steady-state flow

Roadmap - Quasi-One-Dimensional Flow



Chapter 5.3
Area-Velocity Relation

Area-Velocity Relation

$$d(\rho uA) = 0 \Rightarrow uAd\rho + \rho Adu + \rho udA = 0$$

divide by ρuA gives

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Euler's equation:

$$dp = -\rho udu \Rightarrow \frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -udu$$

Assuming adiabatic, reversible (isentropic) process and the definition of speed of sound gives

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial \rho}\right)_s = a^2 \Rightarrow a^2\frac{d\rho}{\rho} = -udu \Rightarrow \frac{d\rho}{\rho} = -M^2\frac{du}{u}$$

Area-Velocity Relation

Now, inserting the expression for $\dfrac{d\rho}{\rho}$ in the rewritten continuity equation gives

$$(1 - M^2)\frac{du}{u} + \frac{dA}{A} = 0$$

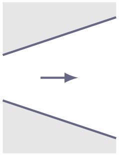
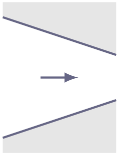
or

$$\frac{dA}{A} = (M^2 - 1)\frac{du}{u}$$

which is the **area-velocity relation**

The Area-Velocity Relation

$$\frac{dA}{A} = \frac{du}{u}(M^2 - 1)$$

		Subsonic $M < 1$	Supersonic $M > 1$
	$dA > 0$	subsonic diffuser $du < 0$ $dp > 0$	supersonic nozzle $du > 0$ $dp < 0$
	$dA < 0$	subsonic nozzle $du > 0$ $dp < 0$	supersonic diffuser $du < 0$ $dp > 0$

The Area-Velocity Relation

$$\frac{du}{u}(M^2 - 1) = \frac{dA}{A}$$

What happens when $M = 1$?

The Area-Velocity Relation

$$\frac{du}{u}(M^2-1)=\frac{dA}{A}$$

What happens when $M=1$?

$M=1$ when $dA=0$

The Area-Velocity Relation

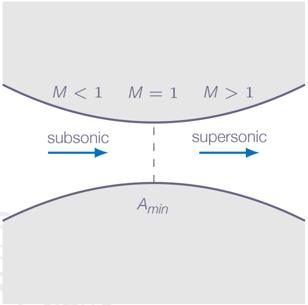
$$\frac{du}{u}(M^2-1)=\frac{dA}{A}$$

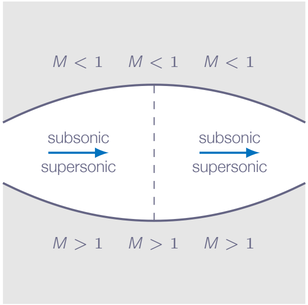
What happens when $M=1$?

$M=1$ when $dA=0$

maximum or minimum area

The Area-Velocity Relation





The Area-Velocity Relation

A converging-diverging nozzle is the **only possibility** to obtain supersonic flow!

A supersonic flow entering a convergent-divergent nozzle will slow down and, if the conditions are right, become sonic at the throat - hard to obtain a shock-free flow in this case

Area-Velocity Relation

$$M \to 0 \Rightarrow \frac{dA}{A} = -\frac{du}{u}$$

$$\frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow$$

$$\frac{1}{Au}\left[udA + Adu\right] = 0 \Rightarrow$$

$$d(uA) = 0 \Rightarrow Au = c$$

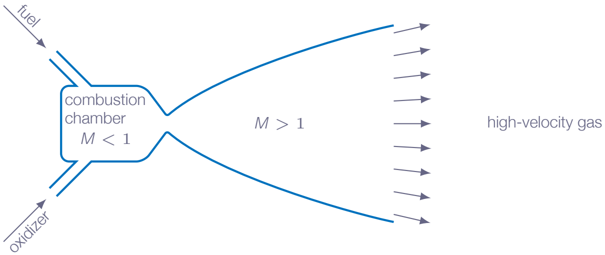
where c is a constant

Area-Velocity Relation

Note 1 The area-velocity relation is only valid for isentropic flow
not valid across a compression shock (due to entropy increase)

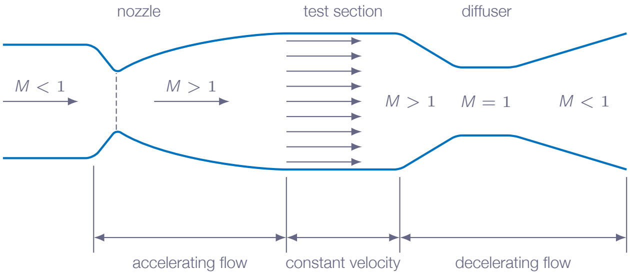
Note 2 The area-velocity relation is valid for all gases

Area-Velocity Relation Examples - Rocket Engine

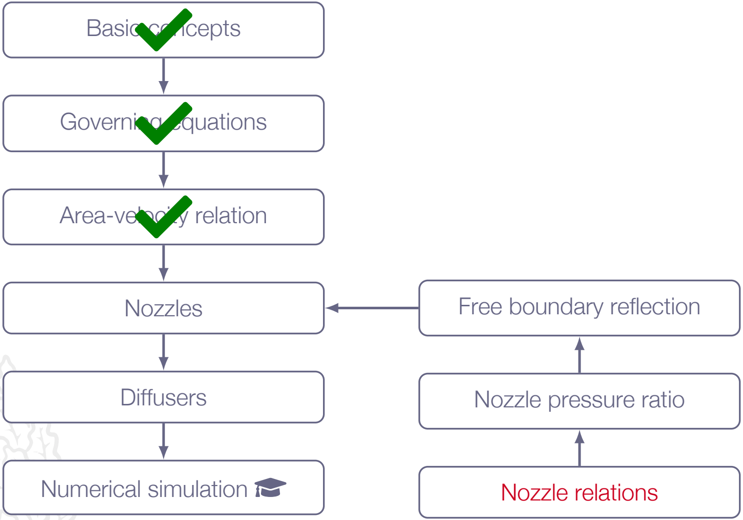


High-temperature, high-pressure gas in combustion chamber expand through the nozzle to very high velocities. Typical figures for a LH^2/LOx rocket engine: $p_o \sim 120$ [bar], $T_o \sim 3600$ [K], exit velocity ~ 4000 [m/s]

Area-Velocity Relation Examples - Wind Tunnel



Roadmap - Quasi-One-Dimensional Flow



Chapter 5.4
Nozzles

Nozzle Flow with Varying Pressure Ratio

time for rocket science!

Nozzle Flow - Relations

Calorically perfect gas assumed:

From Chapter 3:

$$\frac{T_o}{T} = \left(\frac{a_o}{a}\right)^2 = 1 + \frac{1}{2}(\gamma - 1)M^2$$

$$\frac{p_o}{p} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho} = \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

Critical conditions:

$$\frac{T_o}{T^*} = \left(\frac{a_o}{a^*}\right)^2 = \frac{1}{2}(\gamma + 1)$$

$$\frac{p_o}{p^*} = \left(\frac{T_o}{T^*}\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_o}{\rho^*} = \left(\frac{T_o}{T^*}\right)^{\frac{1}{\gamma-1}}$$

Nozzle Flow - Relations

$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a^{*2}} = \frac{u^2}{a^2}\frac{a^2}{a_o^2}\frac{a_o^2}{a^{*2}} \Rightarrow$$

$$\left.\begin{aligned} \frac{u^2}{a^2} &= M^2 \\ \frac{a^2}{a_o^2} &= \left[1 + \frac{1}{2}(\gamma - 1)M^2\right]^{-1} \\ \frac{a_o^2}{a^{*2}} &= \frac{1}{2}(\gamma + 1) \end{aligned}\right\} \Rightarrow M^{*2} = M^2 \frac{\frac{1}{2}(\gamma + 1)}{1 + \frac{1}{2}(\gamma - 1)M^2}$$

Nozzle Flow - Relations

For nozzle flow we have

$$\rho u A = c$$

where c is a constant and therefore

$$\rho^* u^* A^* = \rho u A$$

or, since at critical conditions $u^* = a^*$

$$\rho^* a^* A^* = \rho u A$$

which gives

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{u} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

Nozzle Flow - Relations

$$\frac{A}{A^*} = \frac{\rho^*}{\rho_o}\frac{\rho_o}{\rho}\frac{a^*}{u}$$

$$\left.\begin{aligned}
\frac{\rho^*}{\rho_o} &= \left(\frac{T_o}{T^*}\right)^{\frac{-1}{\gamma-1}} \\
\frac{\rho_o}{\rho} &= \left(\frac{T_o}{T}\right)^{\frac{1}{\gamma-1}} \\
\frac{a^*}{u} &= \frac{1}{M^*}
\end{aligned}\right\} \Rightarrow \frac{A}{A^*} = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{1}{\gamma-1}} M^*}$$

Nozzle Flow - Relations

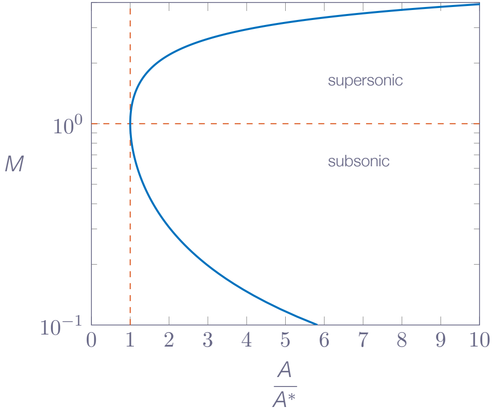
$$\left.\begin{aligned}
\left(\frac{A}{A^*}\right)^2 &= \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{2}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{2}{\gamma-1}} {M^*}^2} \\
{M^*}^2 &= M^2\frac{\frac{1}{2}(\gamma+1)}{1+\frac{1}{2}(\gamma-1)M^2}
\end{aligned}\right\} \Rightarrow$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{\frac{\gamma+1}{\gamma-1}}}{\left[\frac{1}{2}(\gamma+1)\right]^{\frac{\gamma+1}{\gamma-1}} M^2}$$

which is the **area-Mach-number relation**

The Area-Mach-Number Relation

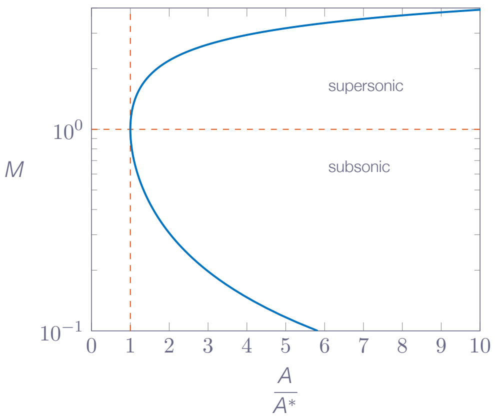
$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$



The Area-Mach-Number Relation

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2+(\gamma-1)M^2}{\gamma+1}\right]^{(\gamma+1)/(\gamma-1)}$$

Note! $\frac{A}{A^*} = \frac{\rho^* u^*}{\rho u}$



Area-Mach-Number Relation

Note 1 Critical conditions used here are those corresponding to **isentropic flow**. Do not confuse these with the conditions in the cases of one-dimensional flow with heat addition and friction

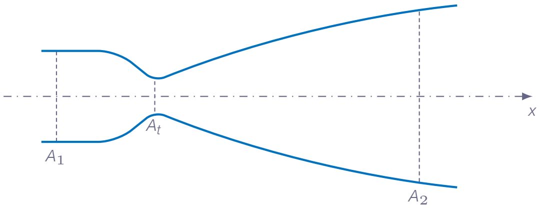
Note 2 For quasi-one-dimensional flow, assuming inviscid steady-state flow, both **total and critical conditions are constant along streamlines** unless shocks are present (then the flow is no longer isentropic)

Note 3 The derived area-Mach-number relation is **only valid for calorically perfect gas and for isentropic flow**. It is not valid across a compression shock

Nozzle Flow

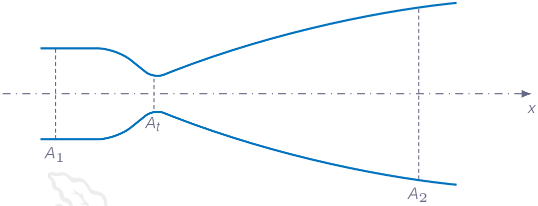
Assumptions:

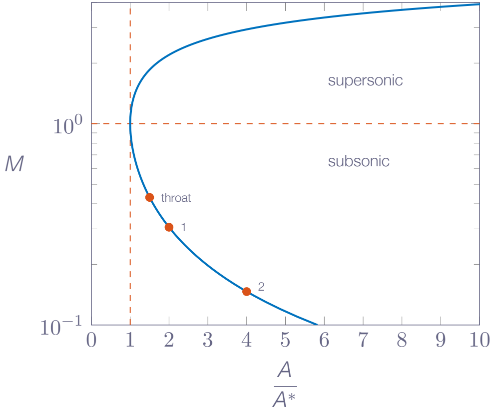
1. inviscid
2. steady-state
3. quasi-one-dimensional
4. calorically perfect gas



The Area-Mach-Number Relation

Sub-critical (non-choked) nozzle flow





$M < 1$ at nozzle throat

$A_t > A^*$

$M_1 < 1$

$M_2 < 1$

The Area-Mach-Number Relation

Subcritical nozzle flow (non-choked and subsonic $\Rightarrow$ isentropic):

A^* is constant throughout the nozzle ($A^* < A_t$)

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_1^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

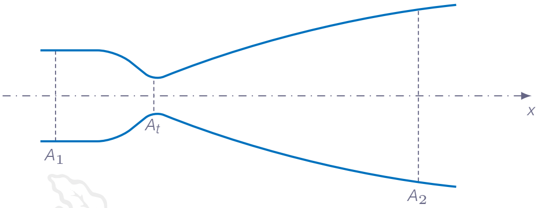
M_2 given by the subsonic solution of

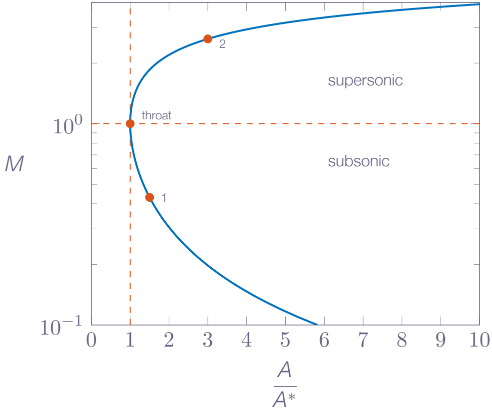
$$\left(\frac{A_2}{A^*}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_2^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M is uniquely determined everywhere in the nozzle, with subsonic flow both upstream and downstream of the throat

The Area-Mach-Number Relation

Critical (choked) nozzle flow





$M = 1$ at nozzle throat

$A_t = A^*$

$M_1 < 1$

$M_2 > 1$

The Area-Mach-Number Relation

Supercritical nozzle flow (choked flow without shocks $\Rightarrow$ isentropic):

A^* is constant throughout the nozzle ($A^* = A_t$)

M_1 given by the subsonic solution of

$$\left(\frac{A_1}{A^*}\right)^2 = \left(\frac{A_1}{A_t}\right)^2 = \frac{1}{M_1^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_1^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

M_2 given by the supersonic solution of

$$\left(\frac{A_2}{A^*}\right)^2 = \left(\frac{A_2}{A_t}\right)^2 = \frac{1}{M_2^2}\left[\frac{2}{\gamma+1}(1+\frac{1}{2}(\gamma-1)M_2^2)\right]^{\frac{\gamma+1}{\gamma-1}}$$

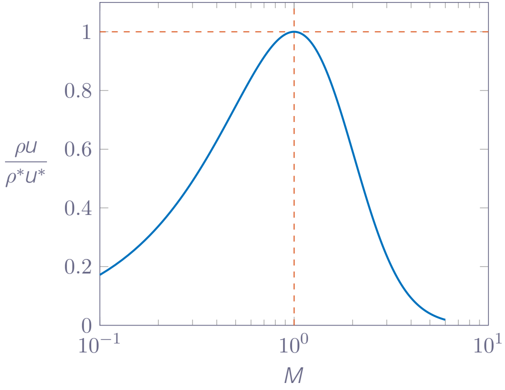
M is uniquely determined everywhere in the nozzle, with subsonic flow upstream of the throat and supersonic flow downstream of the throat

Nozzle Mass Flow

$$\rho u A = \rho^* A^* u^* \Rightarrow \frac{A^*}{A} = \frac{\rho u}{\rho^* u^*}$$

From the area-Mach-number relation

$$\frac{A^*}{A} = \begin{cases} < 1 & \text{if} \quad M < 1 \\ 1 & \text{if} \quad M = 1 \\ < 1 & \text{if} \quad M > 1 \end{cases}$$



The maximum possible massflow through a duct is achieved when its throat reaches sonic conditions

Nozzle Mass Flow

For a choked nozzle:

$$\dot{m} = \rho_1 u_1 A_1 = \rho^* u^* A^* = \rho_2 u_2 A_2$$

$$\left.\begin{aligned} \rho^* &= \frac{\rho^*}{\rho_o}\rho_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \frac{p_o}{RT_o} \\ a^* &= \frac{a^*}{a_o}a_o = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \sqrt{\gamma RT_o} \end{aligned}\right\} \Rightarrow$$

$$\boxed{\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}$$

Nozzle Mass Flow

$$\boxed{\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma+1}{\gamma-1}}}}$$

The **maximum mass flow** that can be sustained through the nozzle

Valid for quasi-one-dimensional, inviscid, steady-state flow and calorically perfect gas

Note! The massflow formula is valid even if there are shocks present downstream of throat!

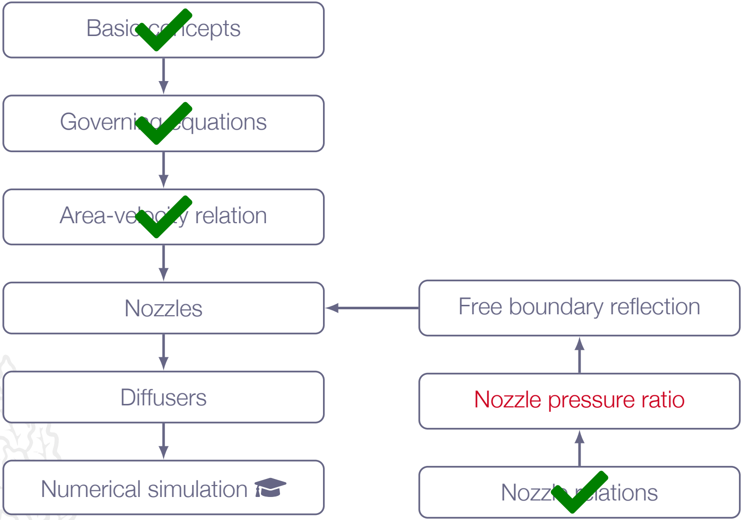
Nozzle Mass Flow

$$\dot{m} = \frac{p_o A_t}{\sqrt{T_o}} \sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

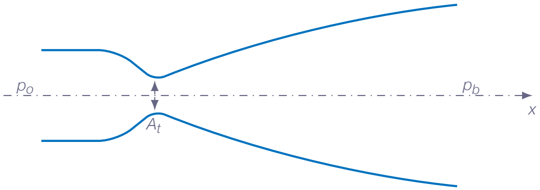
How can we increase mass flow through nozzle?

1. increase p_o
2. decrease T_o
3. increase A_t
4. decrease R
   (increase molecular weight, without changing γ)

Roadmap - Quasi-One-Dimensional Flow

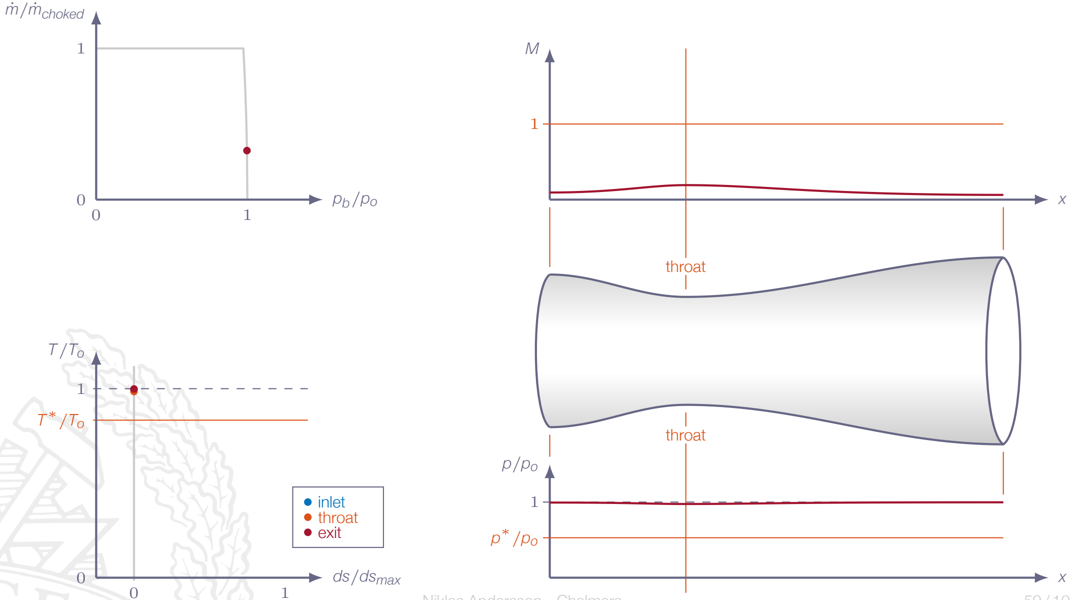


Nozzle Flow with Varying Pressure Ratio

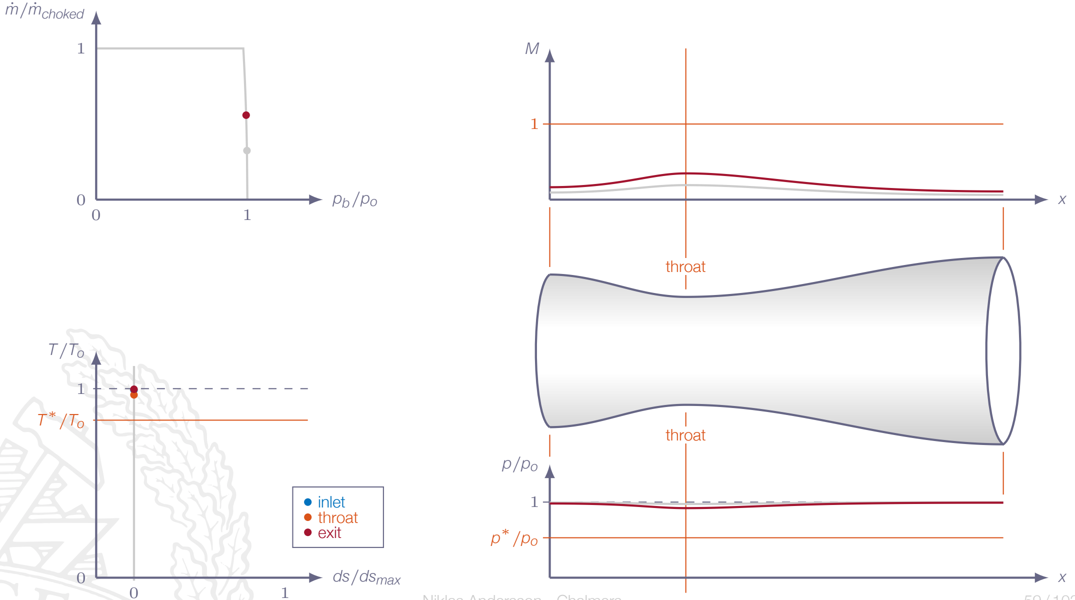


$A(x)$	area function
A_t	$\min\{A(x)\}$
p_o	inlet total pressure
p_b	outlet static pressure (ambient pressure)
p_o/p_b	pressure ratio

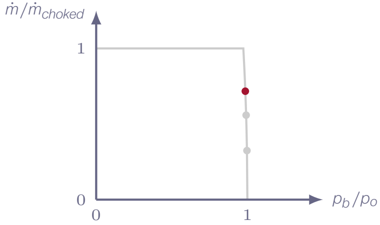
Nozzle Flow with Varying Pressure Ratio

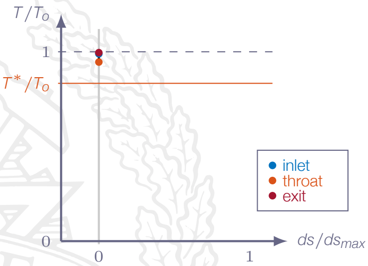


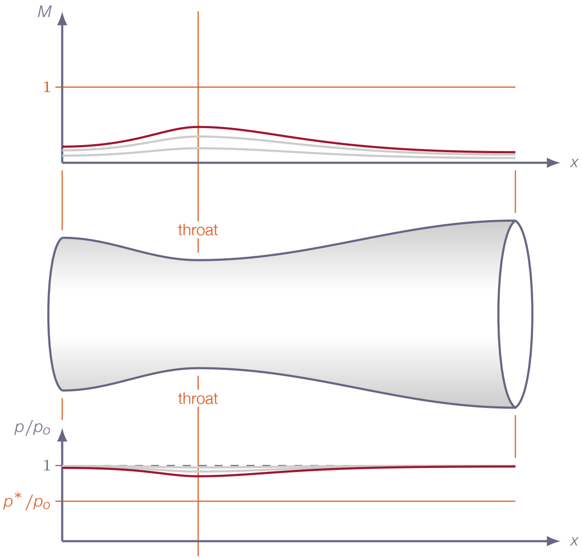
Nozzle Flow with Varying Pressure Ratio



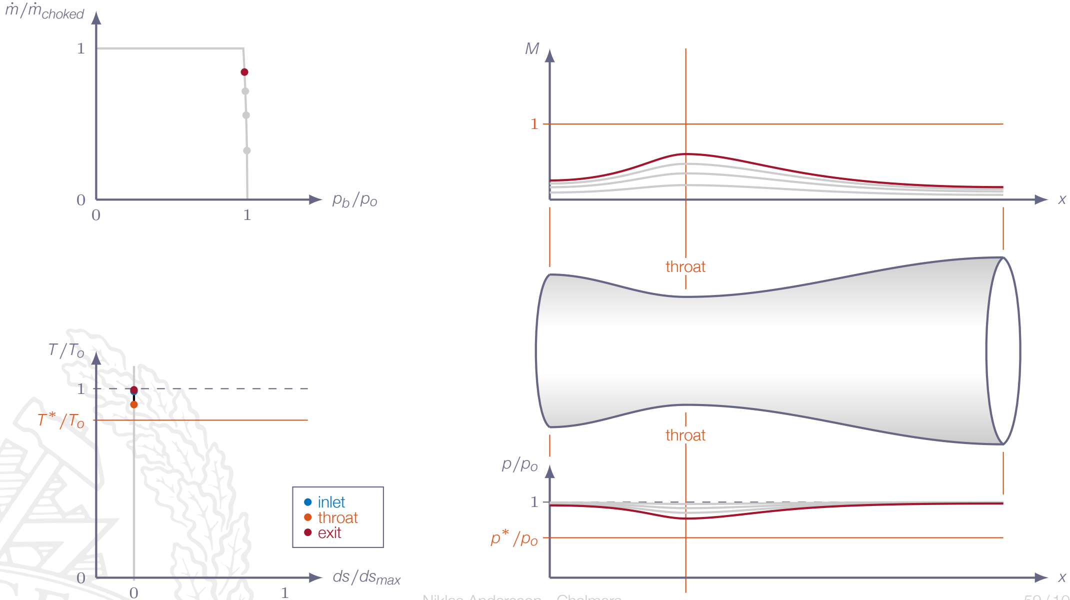
Nozzle Flow with Varying Pressure Ratio



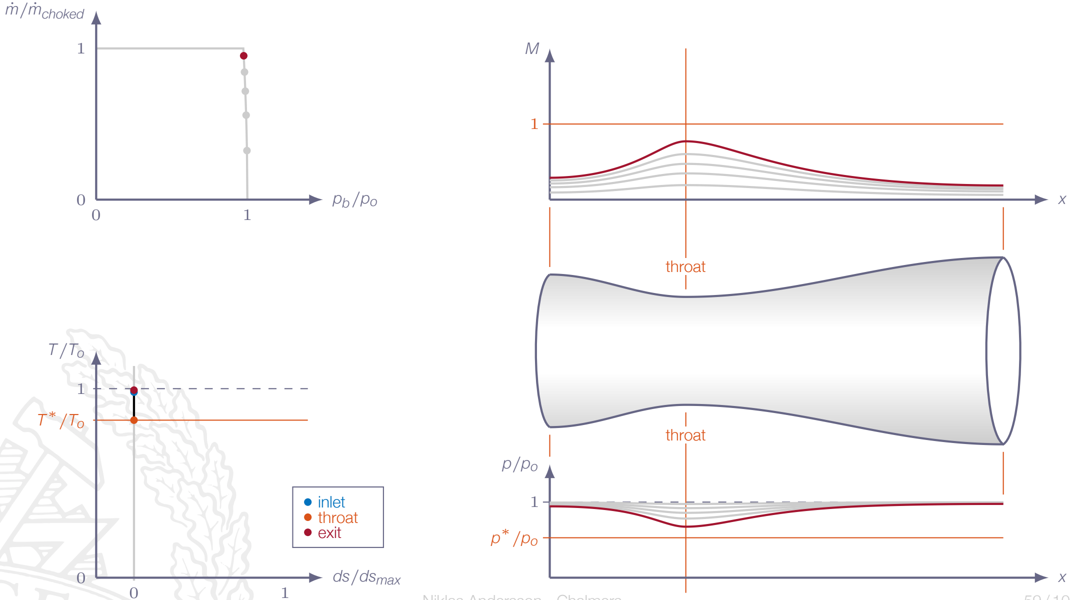




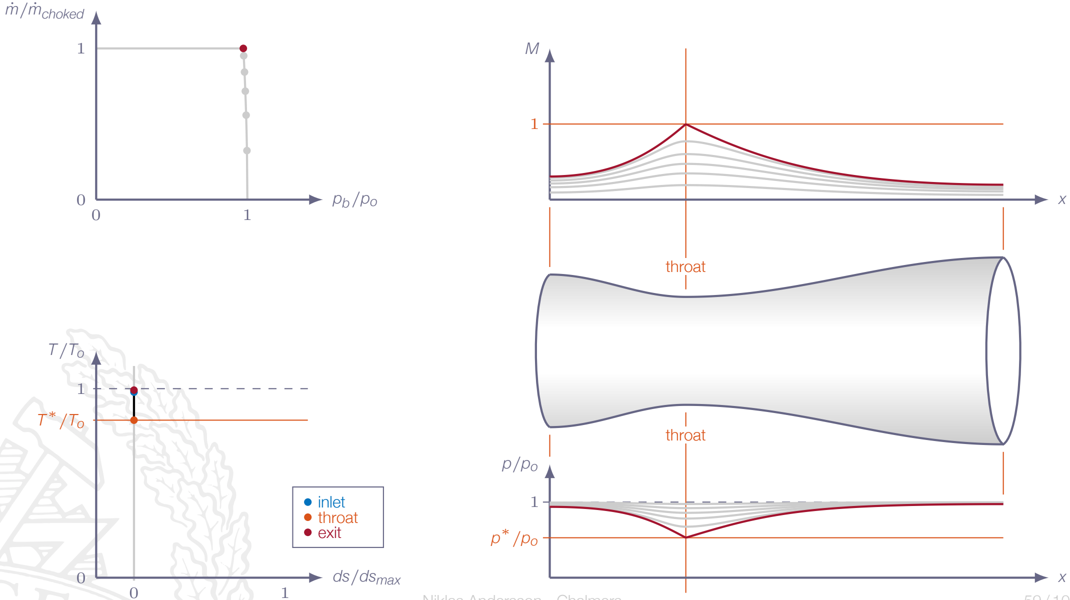
Nozzle Flow with Varying Pressure Ratio



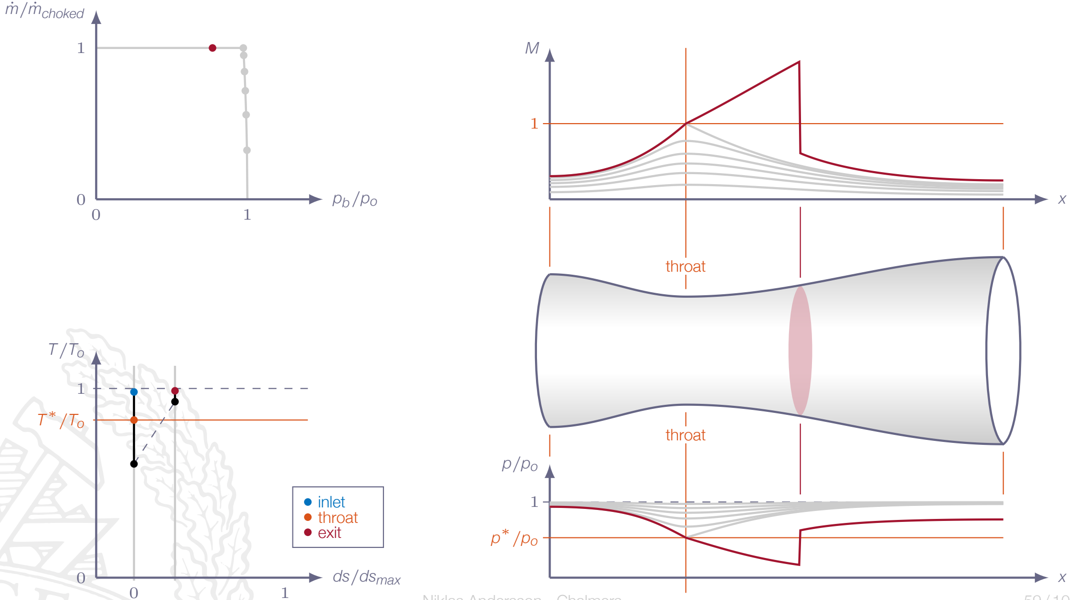
Nozzle Flow with Varying Pressure Ratio



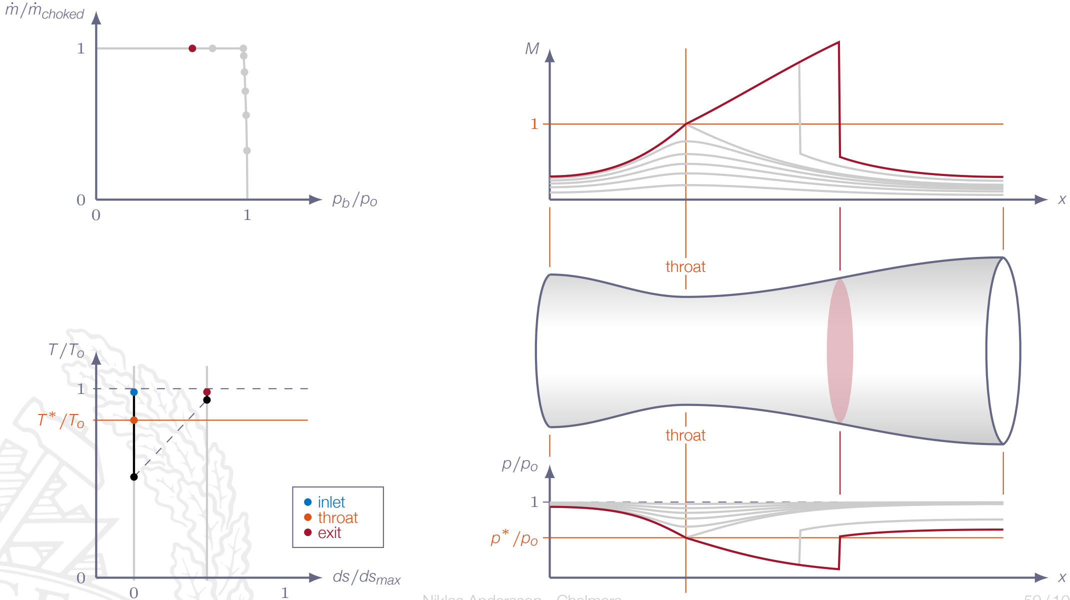
Nozzle Flow with Varying Pressure Ratio



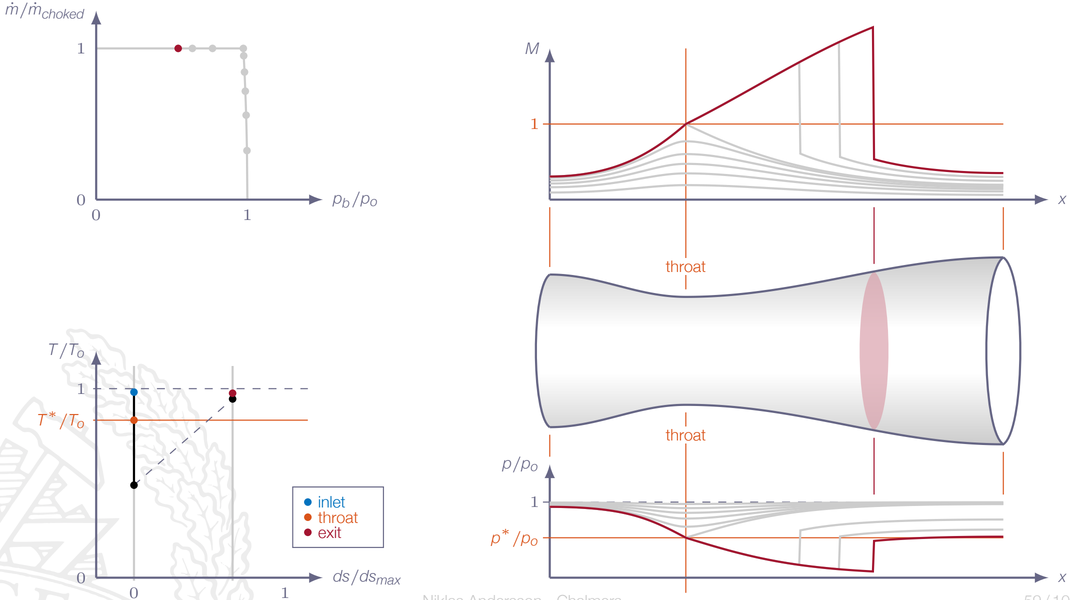
Nozzle Flow with Varying Pressure Ratio



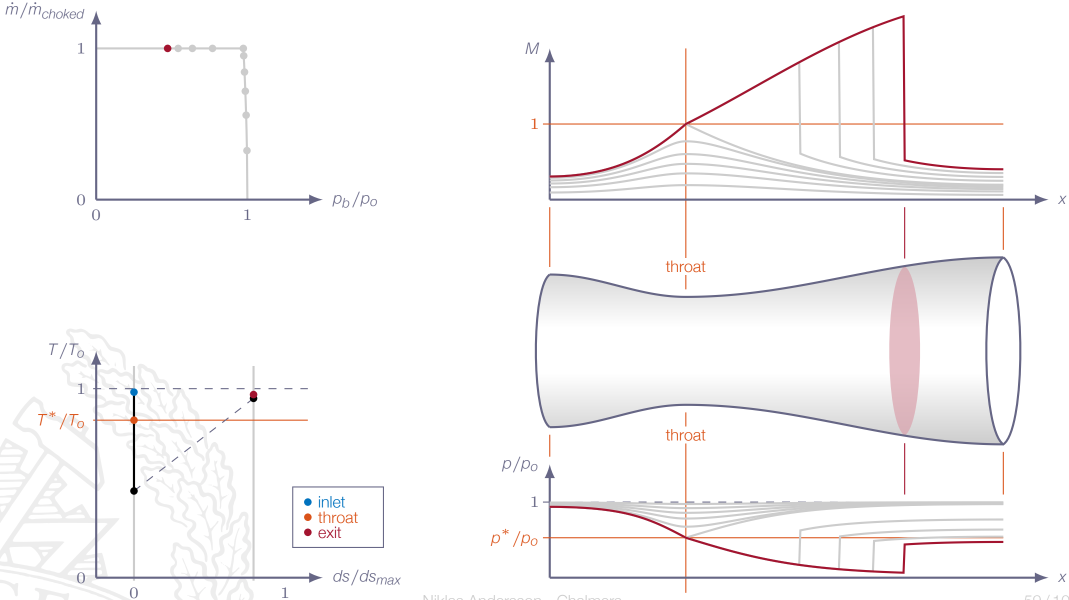
Nozzle Flow with Varying Pressure Ratio



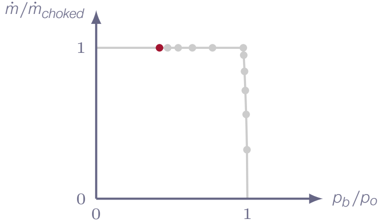
Nozzle Flow with Varying Pressure Ratio

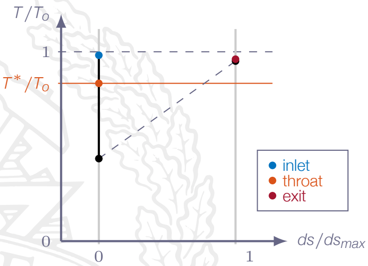


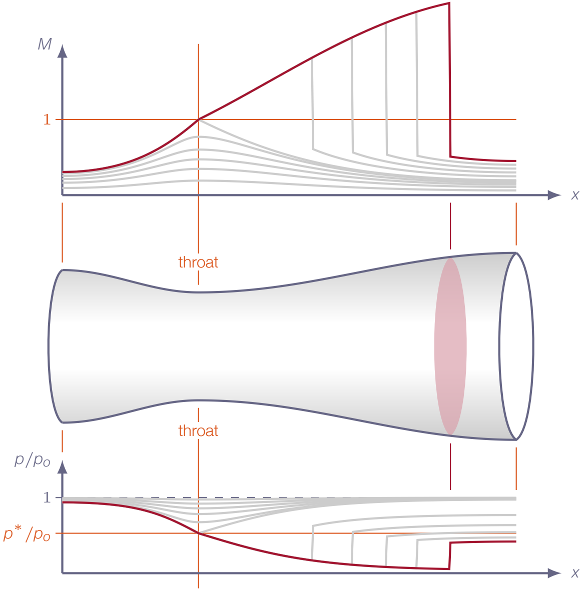
Nozzle Flow with Varying Pressure Ratio



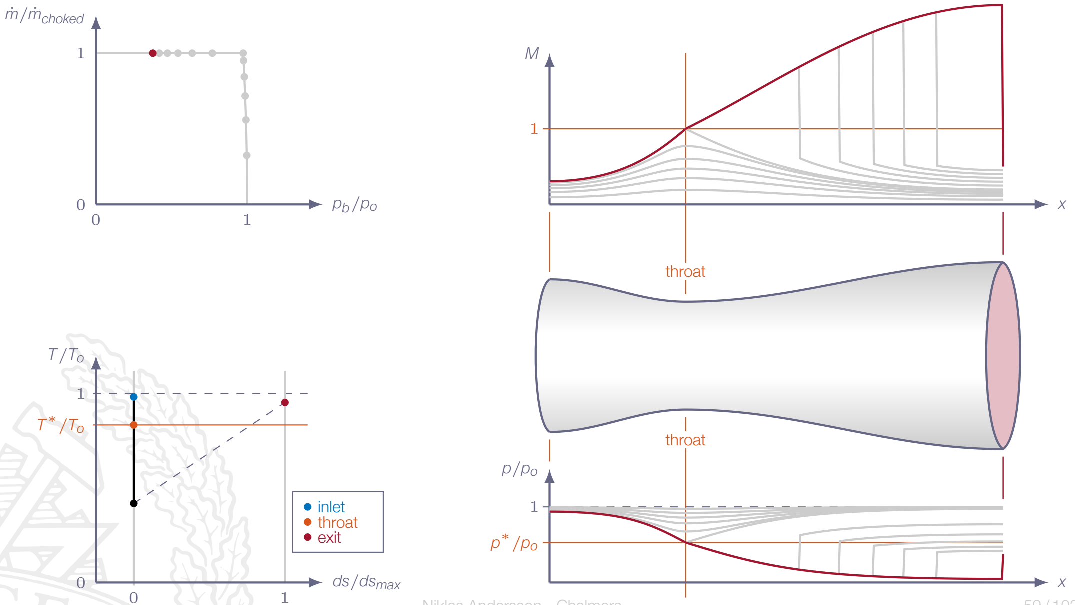
Nozzle Flow with Varying Pressure Ratio



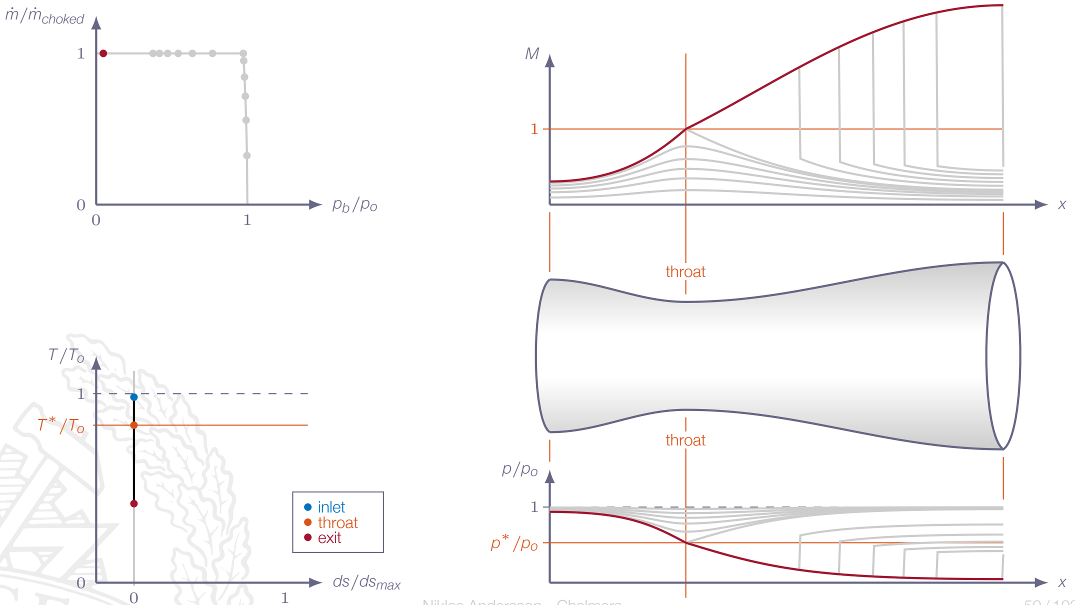




Nozzle Flow with Varying Pressure Ratio



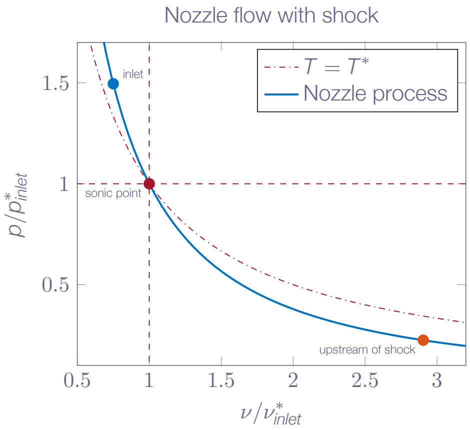
Nozzle Flow with Varying Pressure Ratio



Nozzle Flow with Internal Shock

The nozzle flow process follows an isentrope up to the location of the internal normal shock

Sonic conditions at the nozzle throat

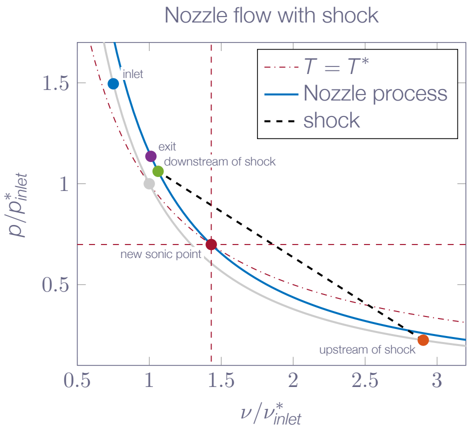


Nozzle Flow with Internal Shock

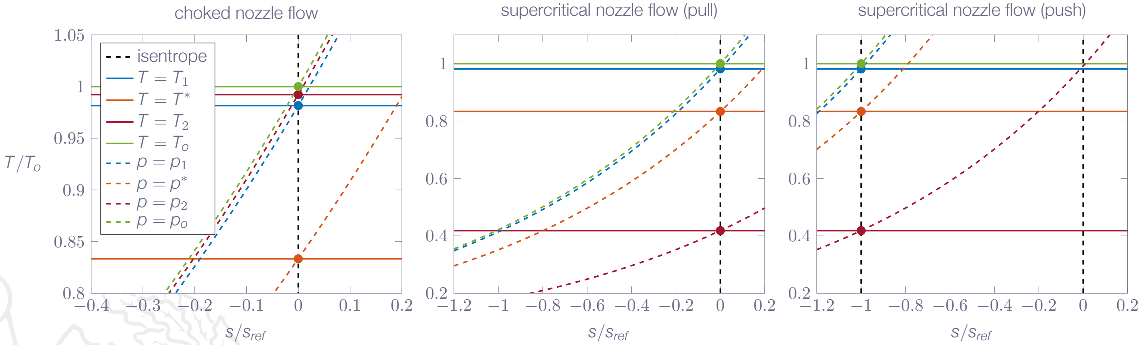
The normal shock moves the process line to another isentrope

T_o and thus T^* is not affected by the shock

p_o decreases over the shock which means that p^* decreases and ν^* increases

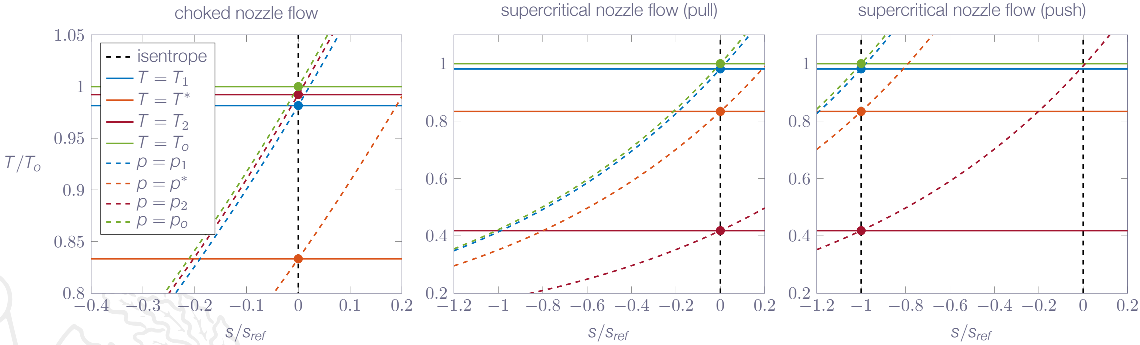


Nozzle Operation - Pull vs. Push



- Nozzle Pressure Ratio $NPR = p_o/p_b$
- Pull - increase NPR by reducing the back pressure (p_b)
- Push - increase NPR by increasing the inlet total pressure (p_o)

Nozzle Operation - Pull vs. Push

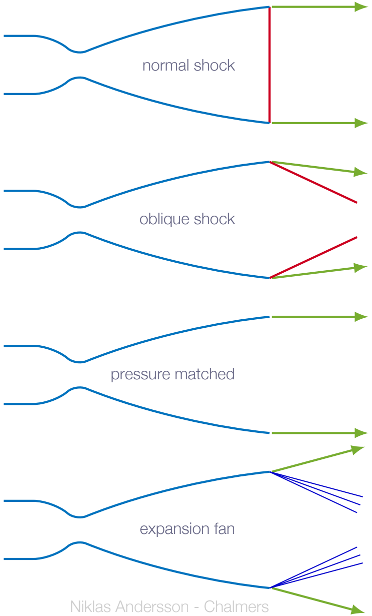


$$\left.\begin{array}{l} p^*_{push} > p^*_{pull} \\ T^*_{push} = T^*_{pull} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \rho^*_{push} > \rho^*_{pull} \\ a^*_{push} = a^*_{pull} \end{array}\right.$$

$$\dot{m} = \rho^* a^* A^* \; (A^*_{push} = A^*_{pull})$$

$$\dot{m}_{push} > \dot{m}_{pull} = \dot{m}_{choked}$$

Nozzle Flow with Varying Pressure Ratio - Downstream Flow



$p_o/p_b = (p_o/p_b)_{ne}$
normal shock at nozzle exit

$(p_o/p_b)_{ne} < p_o/p_b < (p_o/p_b)_{sc}$
overexpanded nozzle flow

$p_o/p_b = (p_o/p_b)_{sc}$
pressure matched nozzle flow

$p_o/p_b > (p_o/p_b)_{sc}$
underexpanded nozzle flow

Nozzle Flow with Varying Pressure Ratio (Summary)

$(p_o/p_b) < (p_o/p_b)_{cr}$

- **subsonic, isentropic** flow throughout the nozzle
- the mass flow changes with p_b, *i.e.* the flow is not choked

$(p_o/p_b) = (p_o/p_b)_{cr}$

- **sonic** flow ($M = 1.0$) at the throat
- the flow will flip to the supersonic solution downstream of the throat, for an infinitesimal increase of (p_o/p_b)

$(p_o/p_b)_{cr} < (p_o/p_b) < (p_o/p_b)_{ne}$

- the flow is **choked** (fixed mass flow)
- a **normal shock** will appear downstream of the throat, with strength and position depending on (p_o/p_b)

Nozzle Flow with Varying Pressure Ratio (Summary)

$(p_o/p_b) = (p_o/p_b)_{ne}$

- **normal shock** at the nozzle exit
- **supersonic, isentropic** flow from throat to exit

$(p_o/p_b)_{ne} < (p_o/p_b) < (p_o/p_b)_{sc}$

- **overexpanded** flow (supersonic, isentropic flow from throat to exit)
- **oblique shocks** formed downstream of the nozzle exit

$(p_o/p_b) = (p_o/p_b)_{sc}$

- **supercritical** flow (pressure matched)
- supersonic, isentropic flow from the throat and downstream of the nozzle exit

$(p_o/p_b)_{sc} < (p_o/p_b)$

- **underexpanded** flow (supersonic, isentropic flow from throat to exit)
- **expansion fans** formed downstream of the nozzle exit

Nozzle Flow with Varying Pressure Ratio - Q1D Limitations

Quasi-one-dimensional theory

When the interior normal shock is ”pushed out” through the exit (by increasing (p_o/p_b), *i.e.* lowering the back pressure), it disappears completely.

The flow through the nozzle is then **shock free** (and thus also **isentropic** since we neglect viscosity).

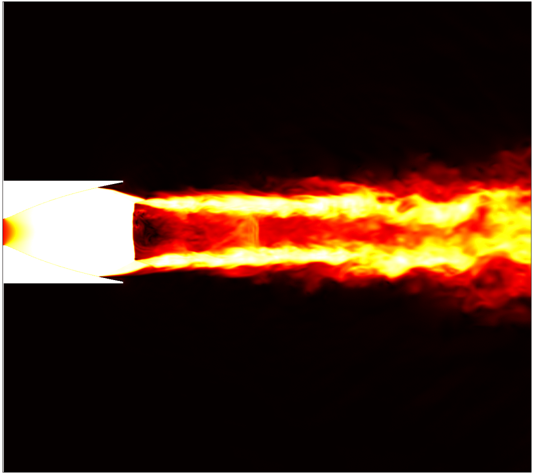
Three-dimensional nozzle flow

When the interior normal shock is ”pushed out” through the exit (by increasing (p_o/p_b)), an **oblique shock** is formed outside of the nozzle exit.

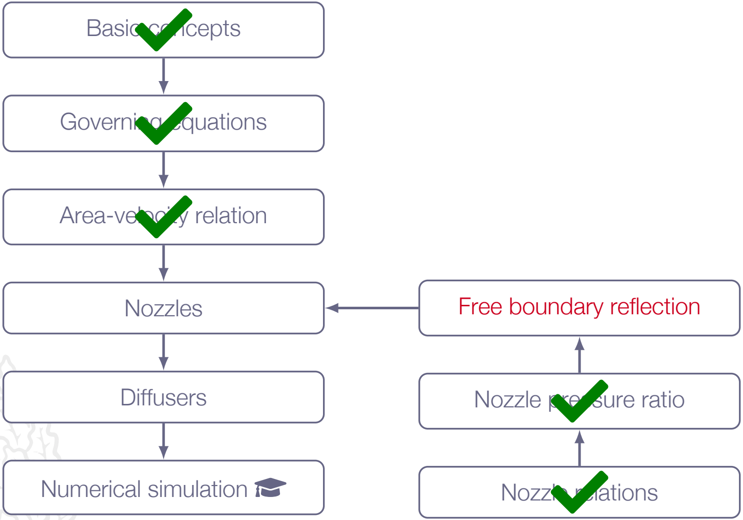
For the exact **supercritical** value of (p_o/p_b) this oblique shock disappears.

For (p_o/p_b) above the supercritical value an **expansion fan** is formed at the nozzle exit.

3D Simulations of Nozzle Flow



Roadmap - Quasi-One-Dimensional Flow

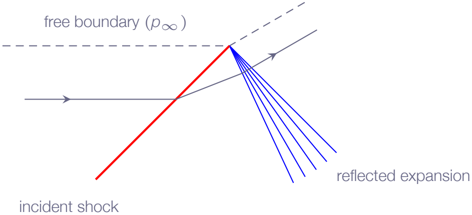


Chapter 5.6
Wave Reflection From a Free Boundary

Free-Boundary Reflection

Free boundary - shear layer, interface between different fluids, etc

Free-Boundary Reflection - Shock Reflection

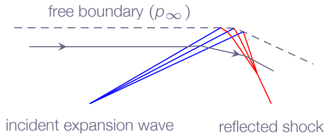


No discontinuity in pressure at the free boundary possible

Incident **shock reflects as expansion** waves at the free boundary

Reflection results in **net turning** of the flow

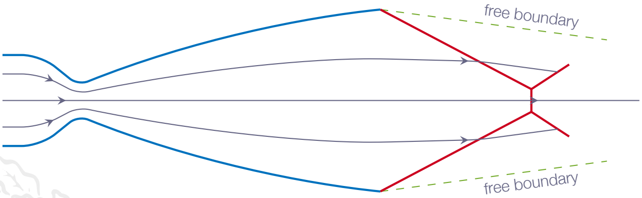
Free-Boundary Reflection - Expansion Wave Reflection



- No discontinuity in pressure at the free boundary possible
- Incident **expansion** waves **reflects as compression** waves at the free boundary
- Finite compression waves coalesces into a shock
- Reflection results in **net turning** of the flow

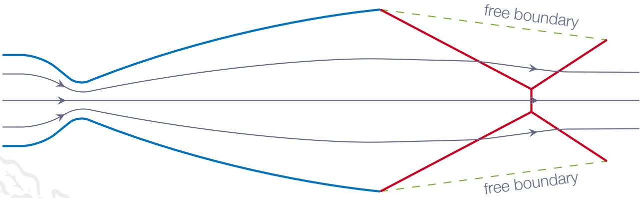
Free-Boundary Reflection - System of Reflections

overexpanded nozzle flow



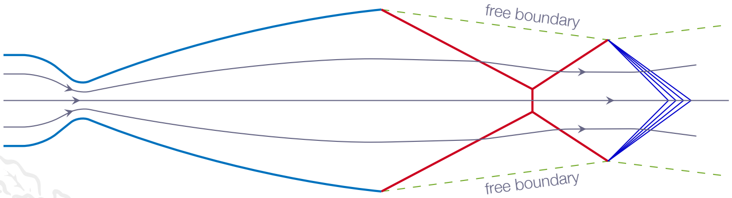
Free-Boundary Reflection - System of Reflections

shock reflection at jet centerline



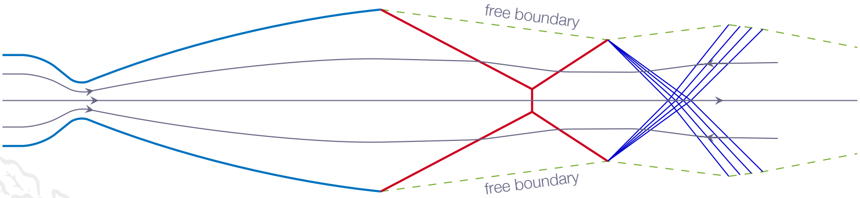
Free-Boundary Reflection - System of Reflections

shock reflection at free boundary



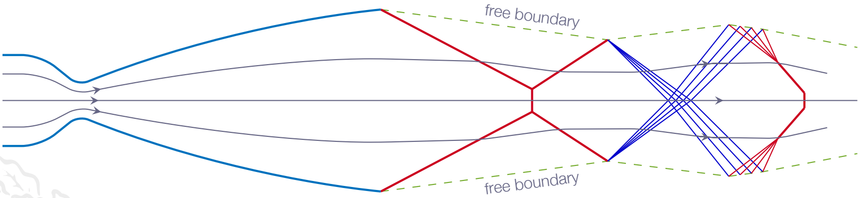
Free-Boundary Reflection - System of Reflections

expansion wave reflection at jet centerline



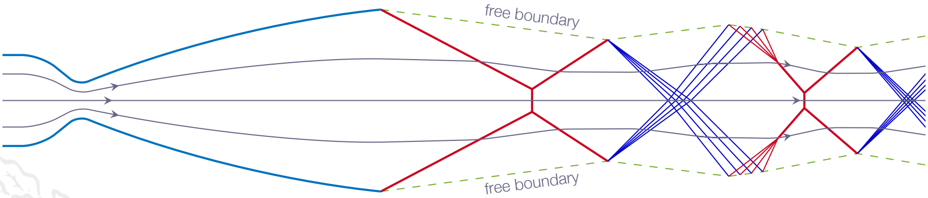
Free-Boundary Reflection - System of Reflections

expansion wave reflection at free boundary



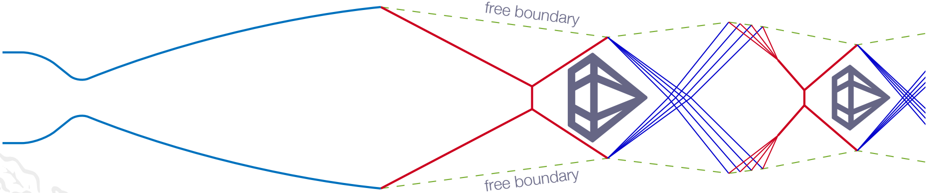
Free-Boundary Reflection - System of Reflections

repeated shock/expansion system



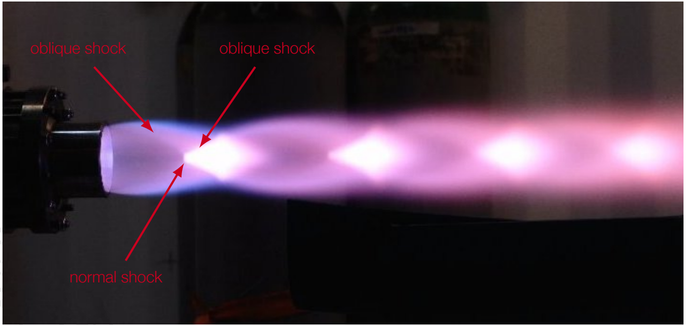
Free-Boundary Reflection - System of Reflections

shock diamonds



Free-Boundary Reflection - System of Reflections

overexpanded jet



Free-Boundary Reflection - Summary

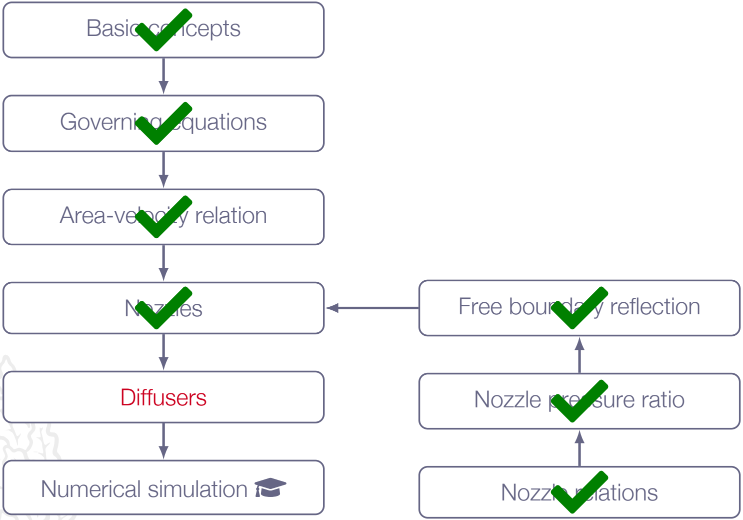
Solid-wall reflection

- Compression waves reflects as compression waves
- Expansion waves reflects as expansion waves

Free-boundary reflection

- Compression waves reflects as expansion waves
- Expansion waves reflects as compression waves

Roadmap - Quasi-One-Dimensional Flow

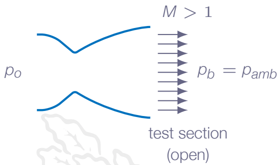


Chapter 5.5
Diffusers

Supersonic Wind Tunnel

wind tunnel with supersonic test section

open test section



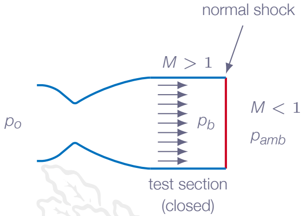
$p_o/p_b = (p_o/p_b)_{sc}$

$M = 3.0$ in test section $\Rightarrow p_o/p_b = 36.7$!!!

Supersonic Wind Tunnel

wind tunnel with supersonic test section

enclosed test section, normal shock at exit



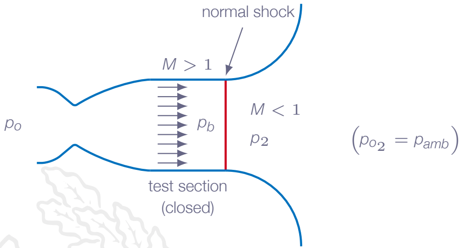
$p_o/p_{amb} = (p_o/p_b)(p_b/p_{amb}) < (p_o/p_b)_{sc}$

$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33 = 3.55$

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add subsonic diffuser after normal shock



$p_o/p_{amb} = (p_o/p_b)(p_b/p_2)(p_2/p_{o_2})$

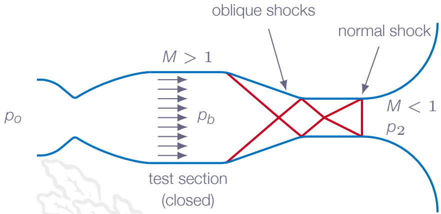
$M = 3.0$ in test section $\Rightarrow$
$p_o/p_{amb} = 36.7/10.33/1.17 = 3.04$

Note! this corresponds exactly to total pressure loss across normal shock

Supersonic Wind Tunnel

wind tunnel with supersonic test section

add supersonic diffuser before normal shock



$\left(p_{o_2} = p_{amb}\right)$

well-designed supersonic + subsonic diffuser $\Rightarrow$

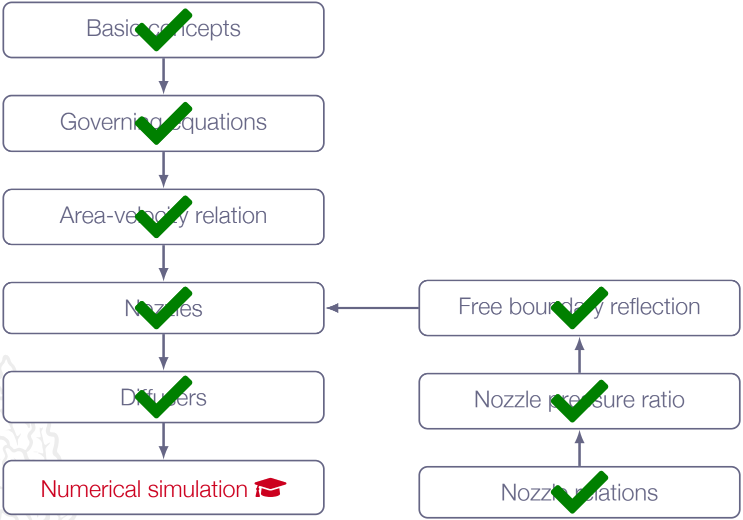
1. decreased total pressure loss
2. decreased p_o and power to drive wind tunnel

Supersonic Wind Tunnel

Main problems:

1. **Complex 3D flow** in the diffuser section
   - viscous effects
   - complex systems of oblique shocks
   - flow separation
   - shock/boundary-layer interaction
2. **Starting requirements**
   second throat must be significantly larger than first throat
   - **variable geometry** diffuser
   - second throat larger during startup procedure
   - decreased second throat to optimum value after supersonic flow is established

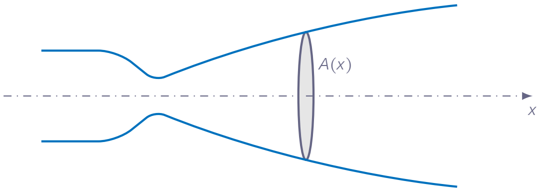
Roadmap - Quasi-One-Dimensional Flow



Quasi-One-Dimensional Euler Equations

Quasi-One-Dimensional Euler Equations

Example: choked flow through a convergent-divergent nozzle



Assumptions: inviscid, $Q = Q(x, t)$

Quasi-One-Dimensional Euler Equations

$$A(x)\frac{\partial}{\partial t}Q + \frac{\partial}{\partial x}\left[A(x)E\right] = A'(x)H$$

where $A(x)$ is the cross section area and

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho e_o \end{bmatrix}, \; E(Q) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho h_o u \end{bmatrix}, \; H(Q) = \begin{bmatrix} 0 \\ p \\ 0 \end{bmatrix}$$

Numerical Approach

Discretization:
Finite-Volume Method (FVM) - Quasi-1D formulation

Numerical scheme:
third-order characteristic upwind scheme

Time stepping technique:
three-stage second-order Runge-Kutta explicit time marching

Boundary conditions:

left-end boundary:

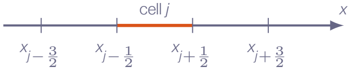
- subsonic inflow
- specify: inlet total temperature (T_o) and total pressure (p_o)

right-end boundary:

- subsonic outflow
- specify: outlet static pressure (p)

Finite-Volume Spatial Discretization

$$\left(\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}\right)$$



Integration over cell j gives:

$$\frac{1}{2}\left[A(x_{j-\frac{1}{2}}) + A(x_{j+\frac{1}{2}})\right]\Delta x_j \frac{d}{dt}\bar{Q}_j +$$
$$\left[A(x_{j+\frac{1}{2}})\hat{E}_{j+\frac{1}{2}} - A(x_{j-\frac{1}{2}})\hat{E}_{j-\frac{1}{2}}\right] =$$
$$\left[A(x_{j+\frac{1}{2}}) - A(x_{j-\frac{1}{2}})\right]\hat{H}_j$$

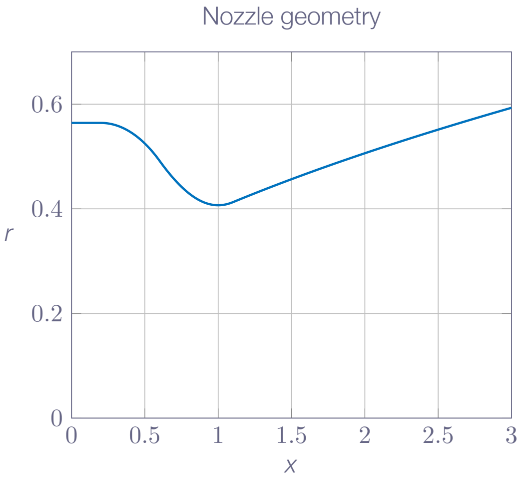
Finite-Volume Spatial Discretization

$$\bar{Q}_j = \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} QA(x)dx\right) \Bigg/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A(x)dx\right)$$

$$\hat{E}_{j+\frac{1}{2}} \approx E\left(Q\left(x_{j+\frac{1}{2}}\right)\right)$$

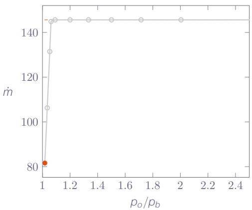
$$\hat{H}_j \approx \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} HA'(x)dx\right) \Bigg/ \left(\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} A'(x)dx\right)$$

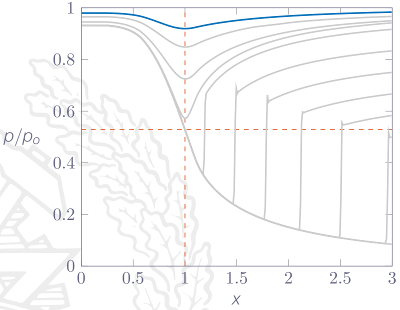
Nozzle Simulation - Back Pressure Sweep

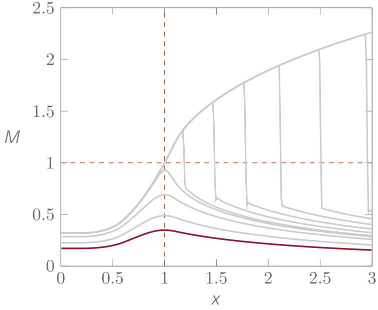


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.18 [bar]
p_o/p_b	1.02
$\dot{m}$	81.61 [kg/s]
M_{max}	0.35

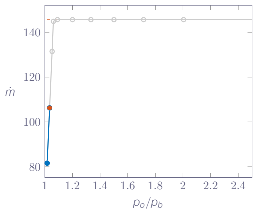


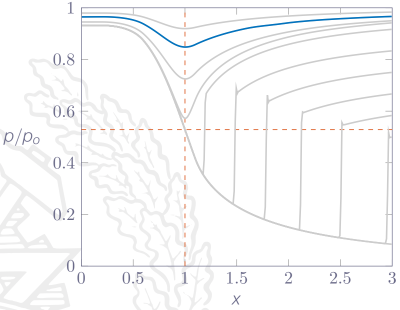


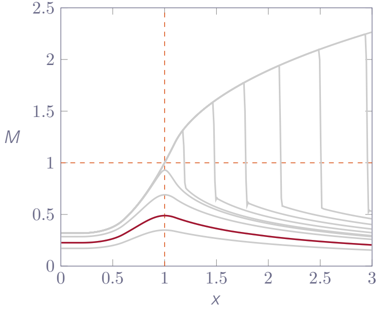


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.16 [bar]
p_o/p_b	1.03
$\dot{m}$	106.27 [kg/s]
M_{max}	0.49

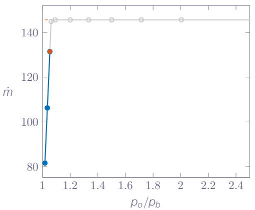


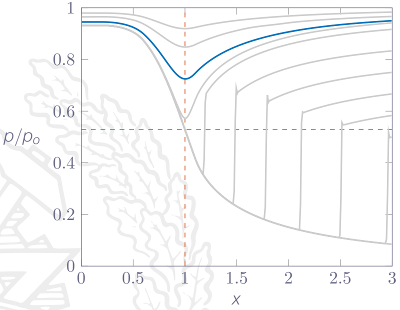


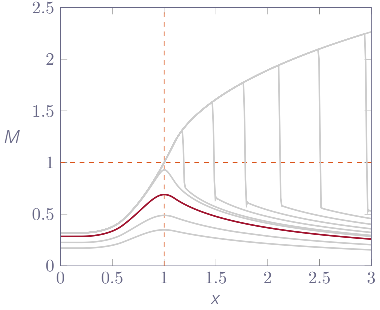


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_b	1.14	[bar]
p_o/p_b	1.05	
$\dot{m}$	131.45	[kg/s]
M_{max}	0.69	

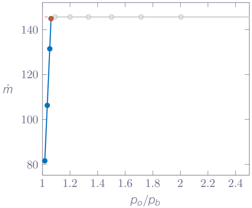


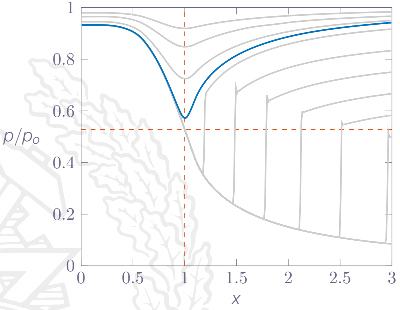


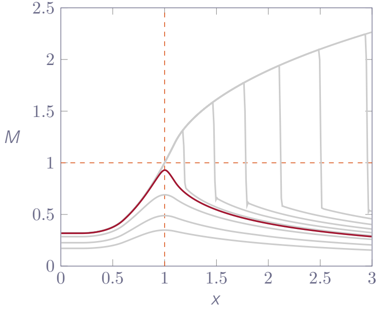


Nozzle Simulation - Back Pressure Sweep

p_o	1.20	[bar]
p_b	1.13	[bar]
p_o/p_b	1.06	
$\dot{m}$	144.88	[kg/s]
M_{max}	0.93	

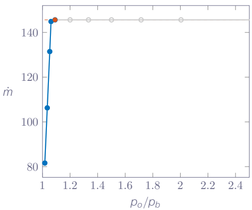


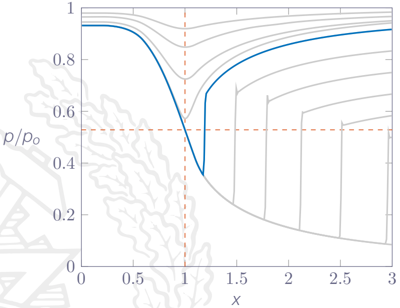


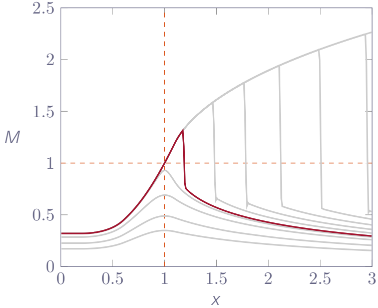


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.10 [bar]
p_o/p_b	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

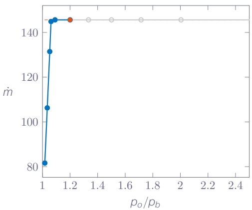


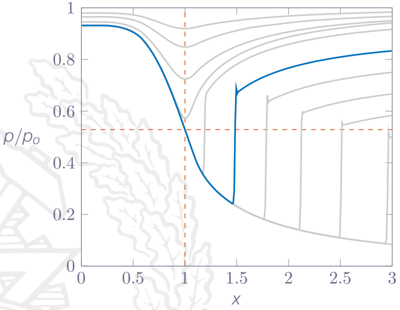


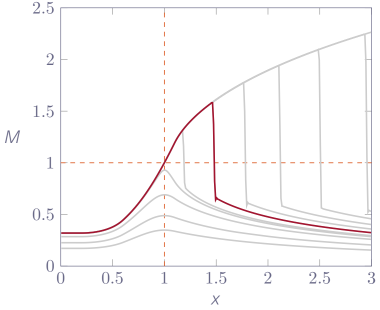


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.00 [bar]
p_o/p_b	1.20
$\dot{m}$	145.6 [kg/s]
M_{max}	1.58

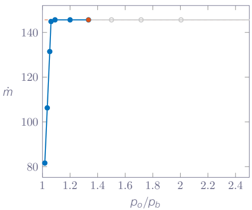


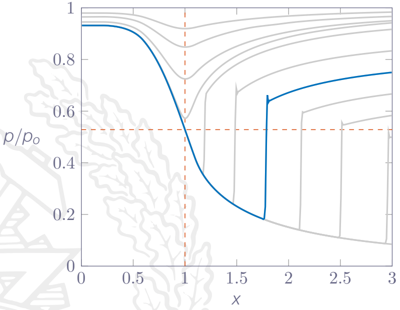


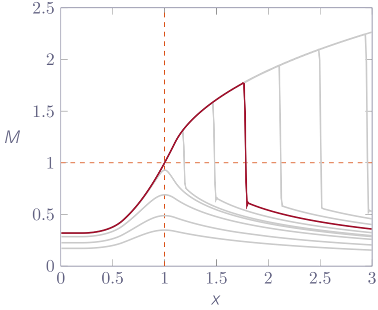


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.90 [bar]
p_o/p_b	1.33
$\dot{m}$	145.6 [kg/s]
M_{max}	1.77

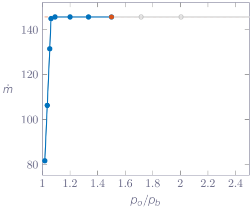


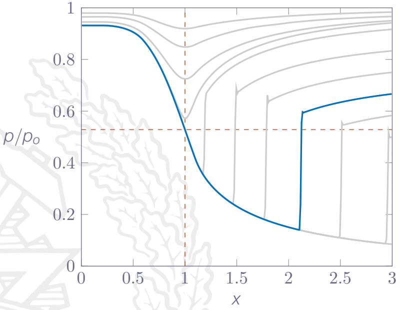


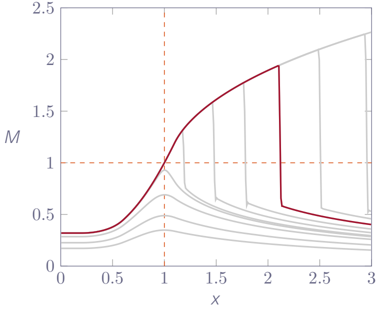


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.80 [bar]
p_o/p_b	1.50
$\dot{m}$	145.6 [kg/s]
M_{max}	1.94

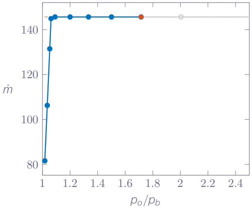


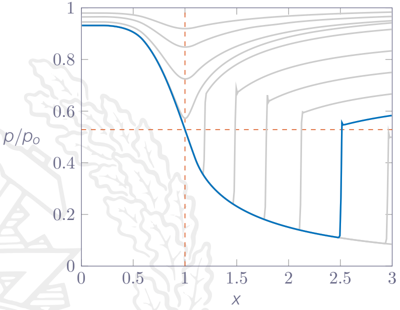


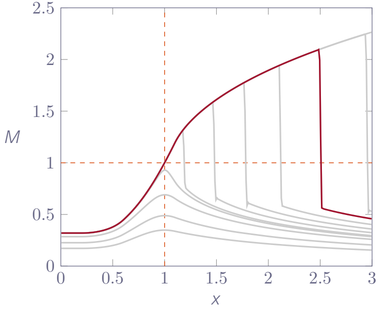


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.70 [bar]
p_o/p_b	1.71
$\dot{m}$	145.6 [kg/s]
M_{max}	2.10

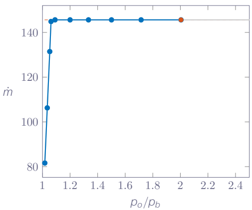


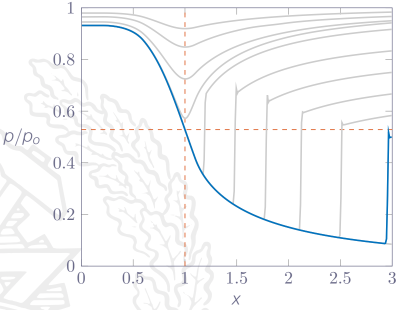


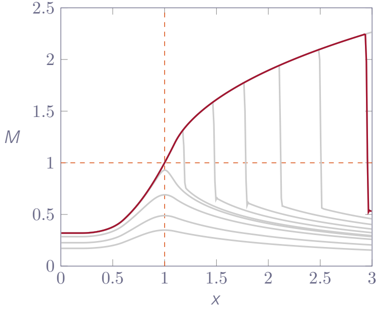


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.60 [bar]
p_o/p_b	2.00
$\dot{m}$	145.6 [kg/s]
M_{max}	2.24

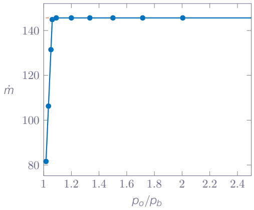


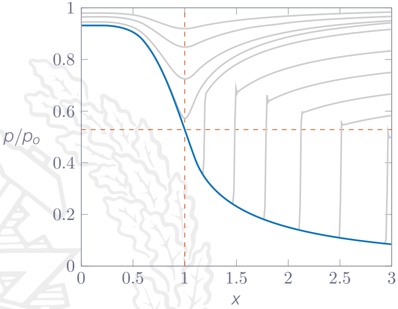


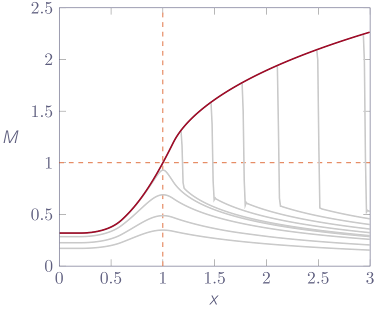


Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	0.50 [bar]
p_o/p_b	11.8
$\dot{m}$	145.6 [kg/s]
M_{max}	2.26

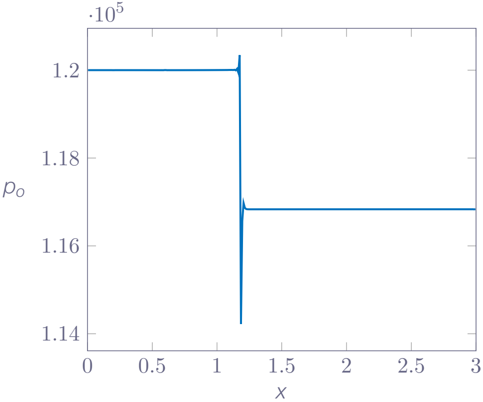






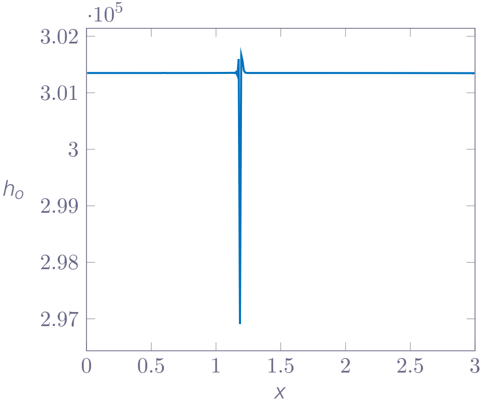
Nozzle Simulation - Back Pressure Sweep

p_o	1.20 [bar]
p_b	1.10 [bar]
p_o/p_b	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31

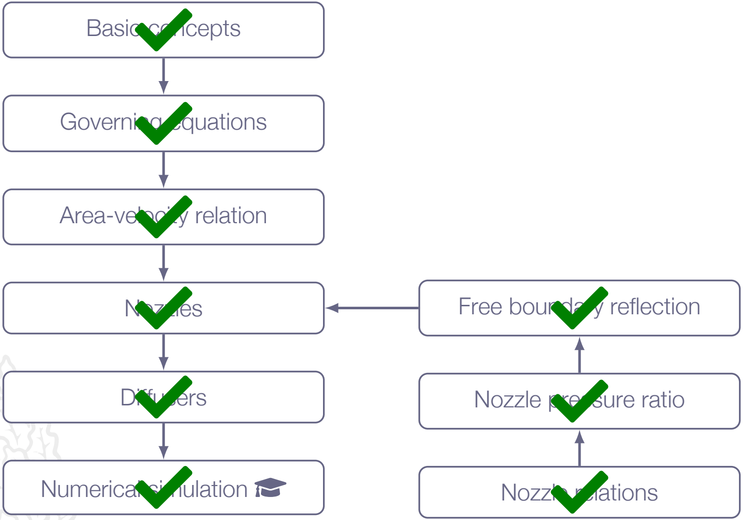


Nozzle Simulation - Back Pressure Sweep

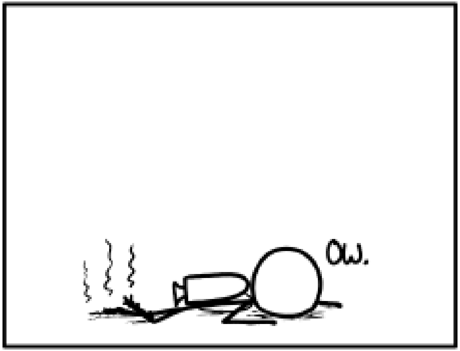
p_o	1.20 [bar]
p_b	1.10 [bar]
p_o/p_b	1.09
$\dot{m}$	145.62 [kg/s]
M_{max}	1.31



Roadmap - Quasi-One-Dimensional Flow



ROCKET PACKS ARE EASY.



THE HARD PART IS INVENTING
THE CALF SHIELDS.