

Compressible Flow - TME085

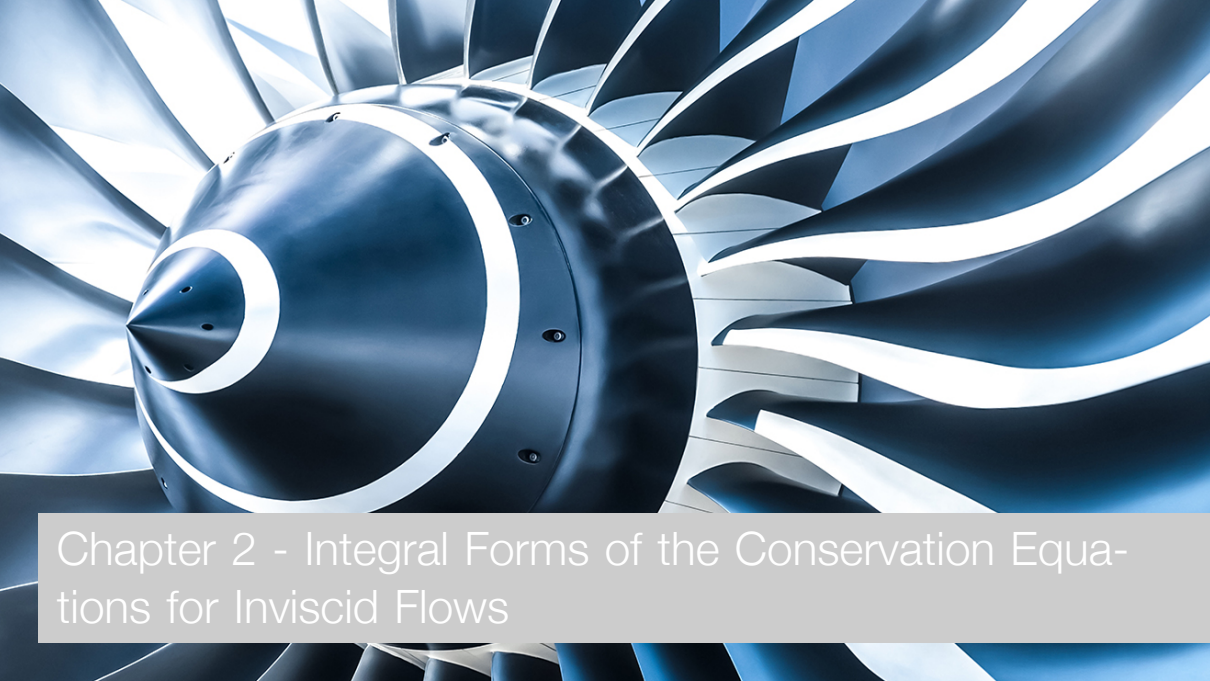
Chapter 2

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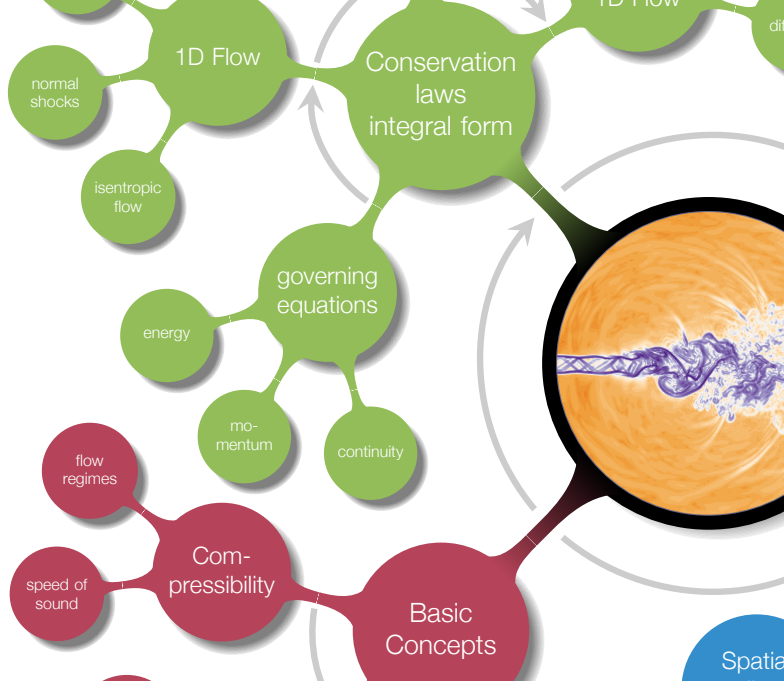
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Chapter 2 - Integral Forms of the Conservation Equations for Inviscid Flows

Overview

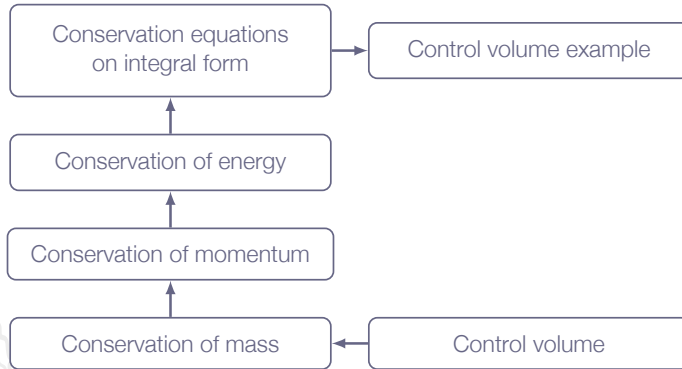


Learning Outcomes

- 4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on
- 5 **Explain** how thermodynamic relations enter into the flow equations
- 7 **Explain** why entropy is important for flow discontinuities

equations, equations and more equations

Roadmap - Integral Relations

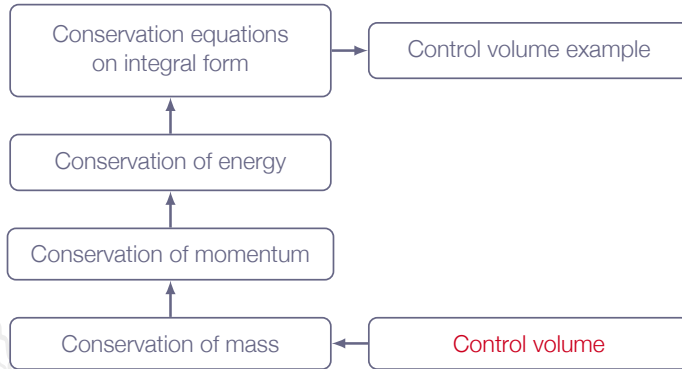


Motivation

We need to formulate the basic form of the governing equations for compressible flow before we get to the applications



Roadmap - Integral Relations



Integral Forms of the Conservation Equations

Conservation principles:

1. conservation of mass
2. conservation of momentum (*Newton's second law*)
3. conservation of energy (*first law of thermodynamics*)



Integral Forms of the Conservation Equations

The control volume approach

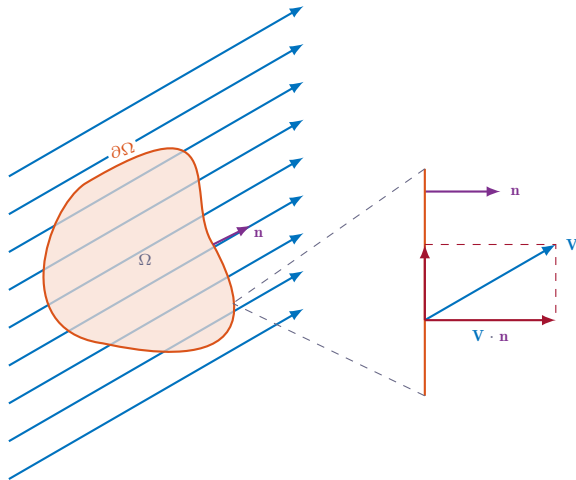
Notation:

Ω fixed control volume

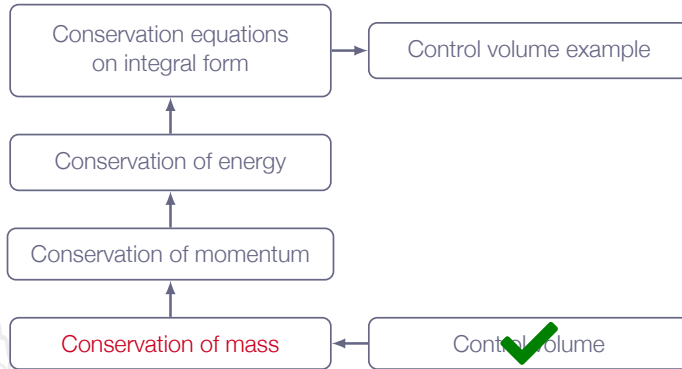
$\partial\Omega$ boundary of Ω

$\mathbf{n}$ outward facing unit normal vector

$\mathbf{v}$ fluid velocity ($v = |\mathbf{v}|$)



Roadmap - Integral Relations



Chapter 2.3

Continuity Equation



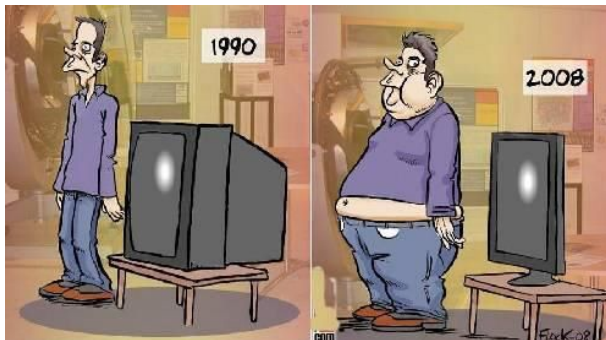
Continuity Equation

Conservation of mass:

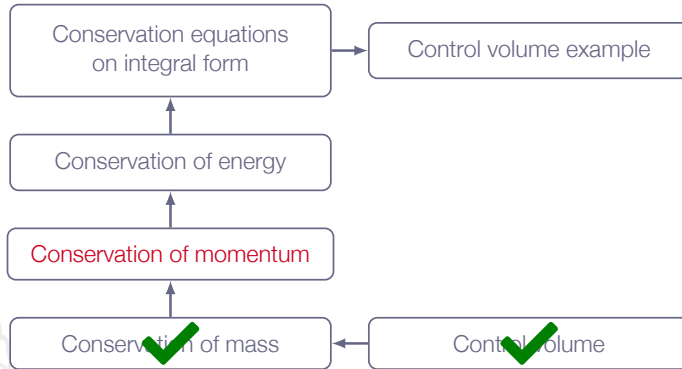
$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho d\mathcal{V}}_{\text{rate of change of total mass in } \Omega} + \underbrace{\oint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{\text{net mass flow out from } \Omega} = 0$$

Note! notation in the text book $\mathbf{n} \cdot d\mathbf{S} = d\mathbf{S}$

Conservation of Mass



Roadmap - Integral Relations



Chapter 2.4

Momentum Equation



Momentum Equation

Conservation of momentum:

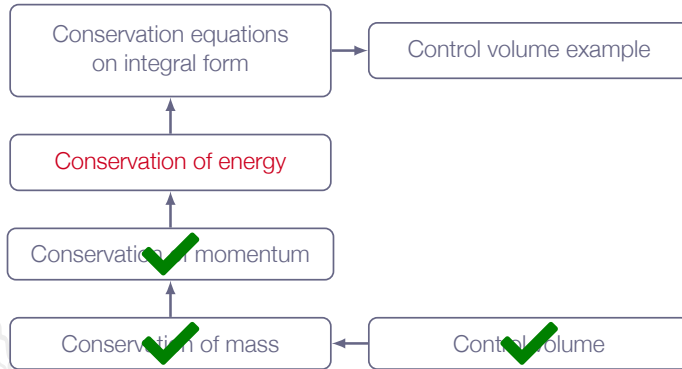
$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho \mathbf{v} d\mathcal{V}}_{\text{rate of change of total momentum in } \Omega} + \underbrace{\oiint_{\partial\Omega} [\rho(\mathbf{v} \cdot \mathbf{n})\mathbf{v} + p\mathbf{n}] dS}_{\text{net momentum flow out from } \Omega \text{ plus surface force on } \partial\Omega \text{ due to pressure}} = \underbrace{\iiint_{\Omega} \rho \mathbf{f} d\mathcal{V}}_{\text{rate of momentum generation due to forces inside } \Omega}$$

Note! friction forces due to viscosity are not included here. To account for these forces, the term $-(\boldsymbol{\tau} \cdot \mathbf{n})$ must be added to the surface integral term. The body force, $\mathbf{f}$, is force per unit mass.

Newton



Roadmap - Integral Relations



Chapter 2.5

Energy Equation



Energy Equation

Conservation of energy:

$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V}}_{\text{rate of change of total internal energy in } \Omega} + \underbrace{\oint_{\partial\Omega} [\rho e_o (\mathbf{v} \cdot \mathbf{n}) + p \mathbf{v} \cdot \mathbf{n}] dS}_{\text{net flow of total internal energy out from } \Omega \text{ plus work due to surface pressure on } \partial\Omega} = \underbrace{\iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathcal{V}}_{\text{work due to forces inside } \Omega}$$

where

$$\rho e_o = \rho \left(e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \right) = \rho \left(e + \frac{1}{2} v^2 \right)$$

is the total internal energy

Energy Equation

The surface integral term may be rewritten as follows:

$$\oiint_{\partial\Omega} \left[\rho \left(e + \frac{1}{2} v^2 \right) (\mathbf{v} \cdot \mathbf{n}) + p \mathbf{v} \cdot \mathbf{n} \right] dS$$

$$\Leftrightarrow$$

$$\oiint_{\partial\Omega} \left[\rho \left(e + \frac{p}{\rho} + \frac{1}{2} v^2 \right) (\mathbf{v} \cdot \mathbf{n}) \right] dS$$

$$\Leftrightarrow$$

$$\oiint_{\partial\Omega} \left[\rho \left(h + \frac{1}{2} v^2 \right) (\mathbf{v} \cdot \mathbf{n}) \right] dS$$

Energy Equation

Introducing total enthalpy

$$h_o = h + \frac{1}{2}v^2$$

we get

$$\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V} + \oint_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n}] dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathcal{V}$$



Energy Equation

Note 1: to include friction work on $\partial\Omega$, the energy equation is extended as

$$\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V} + \oint_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n} - (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{v}] dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathcal{V}$$

Note 2: to include heat transfer on $\partial\Omega$, the energy equation is further extended

$$\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V} + \oint_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n} - (\boldsymbol{\tau} \cdot \mathbf{n}) \cdot \mathbf{v} + \mathbf{q} \cdot \mathbf{n}] dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathcal{V}$$

where $\mathbf{q}$ is the heat flux vector

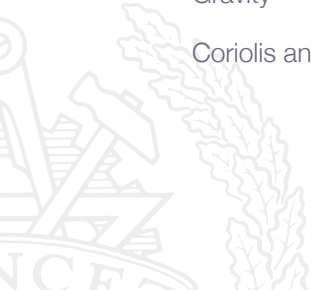
Energy Equation

Note 3: the force $\mathbf{f}$ inside Ω may be a distributed body force field

Examples:

Gravity

Coriolis and centrifugal acceleration terms in a rotating frame of reference



Energy Equation

Note 4: there may be objects inside Ω which we choose to represent as sources of momentum and energy.

For example, there may be a solid object inside Ω which acts on the fluid with a force $\mathbf{F}$ and performs work $\dot{W}$ on the fluid

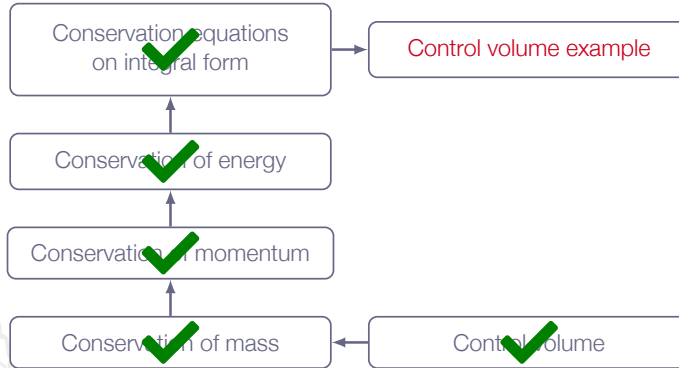
Momentum equation:

$$\frac{d}{dt} \iiint_{\Omega} \rho \mathbf{v} d\mathcal{V} + \iint_{\partial\Omega} [\rho(\mathbf{v} \cdot \mathbf{n})\mathbf{v} + p\mathbf{n}] dS = \iiint_{\Omega} \rho \mathbf{f} d\mathcal{V} + \mathbf{F}$$

Energy equation:

$$\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V} + \iint_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n}] dS = \iiint_{\Omega} \rho \mathbf{f} \cdot \mathbf{v} d\mathcal{V} + \dot{W}$$

Roadmap - Integral Relations



Integral Equations - Applications

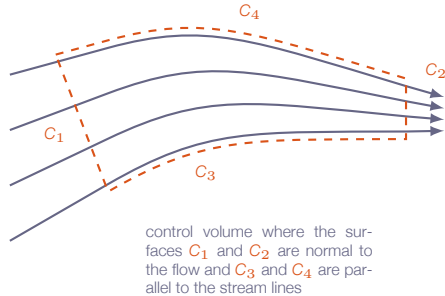
How can we use control volume formulations of conservation laws?

Let $\Omega \rightarrow 0$: In the limit of vanishing volume the control volume formulations give the **P**artial **D**ifferential **E**quations (**PDE**:s) for mass, momentum and energy conservation (see Chapter 6)

Apply in a "smart" way $\Rightarrow$ Analysis tool for many practical problems involving compressible flow (see Chapter 2, Section 2.8)

Integral Equations - Applications

Example: Steady-state adiabatic inviscid flow



Integral Equations - Applications

Conservation of mass:

$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho d\mathcal{V}}_{= 0} + \underbrace{\oiint_{\partial\Omega} \rho \mathbf{v} \cdot \mathbf{n} dS}_{-\rho_1 v_1 A_1 + \rho_2 v_2 A_2} = 0$$

Conservation of energy:

$$\underbrace{\frac{d}{dt} \iiint_{\Omega} \rho e_o d\mathcal{V}}_{= 0} + \underbrace{\oiint_{\partial\Omega} [\rho h_o \mathbf{v} \cdot \mathbf{n}] dS}_{-\rho_1 h_{o1} v_1 A_1 + \rho_2 h_{o2} v_2 A_2} = 0$$

Integral Equations - Applications

Conservation of mass:

$$\rho_1 v_1 A_1 = \rho_2 v_2 A_2$$

Conservation of energy:

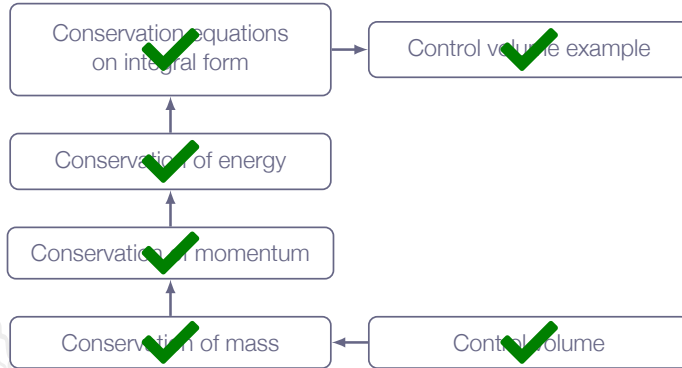
$$\rho_1 h_{o1} v_1 A_1 = \rho_2 h_{o2} v_2 A_2$$

$$\Leftrightarrow$$

$$h_{o1} = h_{o2}$$

Total enthalpy h_o is conserved along streamlines in steady-state adiabatic inviscid flow

Roadmap - Integral Relations



$$E = K_0 t + \frac{1}{2} p v t^2 \quad K_n = \sum_{i=0}^{\infty} \sum_{\pi=0}^{\infty} (n - \pi) (i + e^{\pi - \infty}) \quad \frac{\partial}{\partial t} \nabla \cdot \rho = \frac{8}{23} \oint \rho ds dt \cdot \rho \frac{\partial}{\partial v}$$

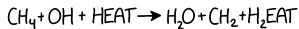
ALL KINEMATICS
EQUATIONS

ALL NUMBER
THEORY EQUATIONS

ALL FLUID DYNAMICS
EQUATIONS

$$|\psi_{x,y}\rangle = A(\psi) A(|x\rangle \otimes |y\rangle)$$

ALL QUANTUM
MECHANICS EQUATIONS



ALL CHEMISTRY
EQUATIONS

$$SU(2) U(1) \times SU(U(2))$$

ALL QUANTUM
GRAVITY EQUATIONS

$$S_g = \frac{-1}{2\epsilon} i \delta \left(\hat{\xi}_0 + P_\epsilon P_v^{abc} \cdot \hat{\eta}_a \right) F_a^\rho \alpha \chi(\xi) \psi(O_a)$$

ALL GAUGE THEORY
EQUATIONS

$$H(t) + \Omega + G \wedge \dots \begin{cases} \dots > 0 & (\text{HUBBLE MODEL}) \\ \dots = 0 & (\text{FLAT SPHERE MODEL}) \\ \dots < 0 & (\text{BRIGHT DARK MATTER MODEL}) \end{cases}$$

ALL COSMOLOGY EQUATIONS

$$\hat{H} - \psi_0 = 0$$

ALL TRULY DEEP
PHYSICS EQUATIONS