Fluid Mechanics MTF053

Equations for Boundary Layer Flows

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1 Introduction

In a boundary layer, the **Navier-Stokes** equations can be simplified by comparing the importance of the individual terms in the equations under conditions representative for boundary-layer flow. This results in a reduced set of equations often referred to as the **boundary-layer equations**. The reduced set of equations is, however, still a system of non-linear partial differential equations and, although it is easier to handle the reduced set of equations than the full Navier-Stokes equations, there are only a limited set of explicit solutions but of course it is also possible to solve the equations numerically. For laminar boundary layers, solutions may be obtained for most geometries whereas for turbulent boundary layers, one often need to use experimental data to be able to solve the equations.

An alternative to using the differential equations is to use relations based on integral equations. These equations are obtained by applying the integral form of the continuity and momentum equations for steady-state flow to a control volume as shown schematically in Fig. 1. The integral relations can be used to obtain approximate solutions for boundary layer flows. For turbulent flows for which, as stated above, experimental data are required to be able to solve the differential equations, the integral relations provide alternative methods that, in some cases, are as accurate as the boundary layer equations.

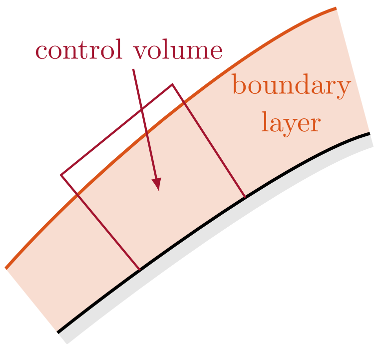

Figure 1: Boundary layer control volume

The boundary-layer equations are derived in section 2 and section 3 is devoted the integral relations mentioned above.

2 The Boundary-Layer Equations

2.1 Laminar Boundary Layers

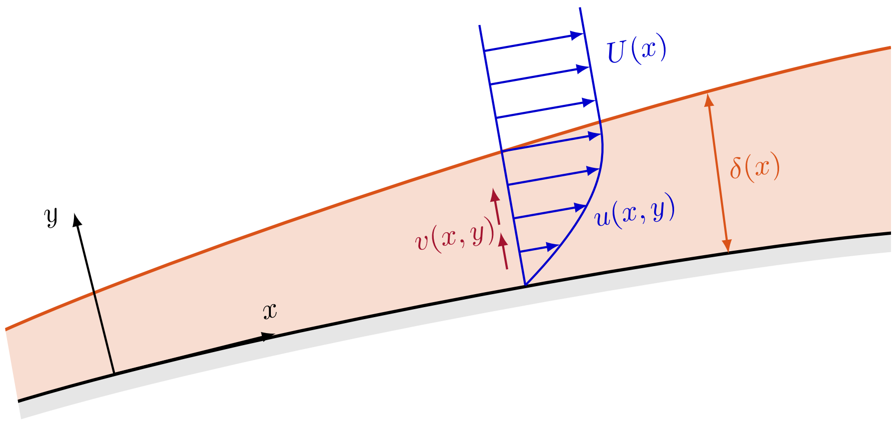

Figure 2: Boundary layer variables and definitions

Let's assume two-dimensional, laminar, incompressible and steady-state flow with negligible body forces. The governing equations for such a flow reads

continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

x-momentum:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{2}$$

y-momentum:

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{3}$$

The first step in the derivation of the boundary-layer equations is to make the governing equations (Eqns. 1-3) non-dimensional. Non-dimensional variables are obtained by scaling with the freestream velocity $U = U(x)$ (see Fig. 2) and a characteristic length L (typically the length of the boundary layer). Note that the asterisks below denote that the variables have been made non-dimensional by scaling.

$$x^* = \frac{x}{L} \qquad u^* = \frac{u}{U}$$

$$y^* = \frac{y}{L} \qquad v^* = \frac{v}{U}$$

$$p^* = \frac{p}{\rho U^2} \qquad Re = \frac{UL}{\nu}$$

If implemented in the continuity equation we get

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial(u^*U)}{\partial(x^*L)} + \frac{\partial(v^*U)}{\partial(y^*L)} = \frac{U}{L}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right) = 0$$

and for the x-momentum equation we get

$$\rho\left(u^*U\frac{\partial(u^*U)}{\partial(x^*L)} + v^*U\frac{\partial(u^*U)}{\partial(y^*L)}\right) = -\frac{\partial(p^*\rho U^2)}{\partial(x^*L)} + \mu\left(\frac{\partial^2(u^*U)}{\partial(x^*L)^2} + \frac{\partial^2(u^*U)}{\partial(y^*L)^2}\right)$$

$$\frac{\rho U^2}{L}\left(u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right) = -\frac{\rho U^2}{L}\frac{\partial p^*}{\partial x^*} + \frac{\mu U}{L^2}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho U L}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

The non-dimensional y-momentum equation is analogous to the x-momentum equation. The final governing equations on non-dimensional form reads.

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{4}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right) \tag{5}$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right) \tag{6}$$

Note the **Reynolds number** in front of the viscous terms in Eqns. 5 and 6 indicating that the viscous terms becomes less important for high-Reynolds number flows.

In order to simplify the governing equations for boundary-layer flows, i.e. deriving the boundary-layer equations, the terms in the non-dimensional equations are compared in terms of magnitude in a boundary-layer context with the aim to find out if some of the terms can be neglected. Thus, we need an estimate of the size of each of the terms given that they are evaluated within a boundary layer. As an example, the magnitude of the y-coordinate is comparable to the boundary-layer thickness δ (see Fig. 2) while the magnitude of the x-coordinate is in the same order as the boundary layer length L.

$$u^* = \frac{u}{U} \sim 1$$

$$x^* = \frac{x}{L} \sim 1$$

$$y^* = \frac{y}{L} \sim \frac{\delta}{L} = \delta^*$$

Note that δ^* is a non-dimensional boundary-layer thickness and not the displacement thickness.

The magnitude of the velocity gradient $\partial u^*/\partial y^*$ is in the same order as that of a line starting from zero at the wall and reaching the freestream velocity U at the edge of the boundary layer $y = \delta$ (see Fig. 3) and thus

$$\frac{\partial u^*}{\partial y^*} \sim \frac{1}{\delta^*}$$

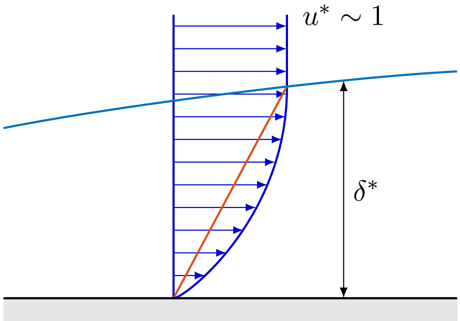

Figure 3: First-order velocity derivative estimate

The value of $\partial u^*/\partial y^*$ changes from $1/\delta^*$ at the wall to zero at the edge of the boundary layer (outside of the boundary layer the velocity is constant and thus $du/dy = 0$) which mean that

$$\frac{\partial^2 u^*}{\partial {y^*}^2} \sim \frac{1}{{\delta^*}^2}$$

With the velocity gradients in the wall-normal direction analyzed, we move on to study the gradients in the flow direction. Evaluating the non-dimensional axial velocity u^* along a line at a constant distance over the wall (see Fig. 4), we see that the value changes from $u^* = 1$ at the left end, where the line is outside of the boundary layer and thus the axial velocity equals the freestream velocity, to zero at the right end, where the line is located within the boundary layer and thus the velocity gets closer to zero the further downstream we move.

$$\frac{\partial u^*}{\partial x^*} \sim \frac{1}{1} = 1$$

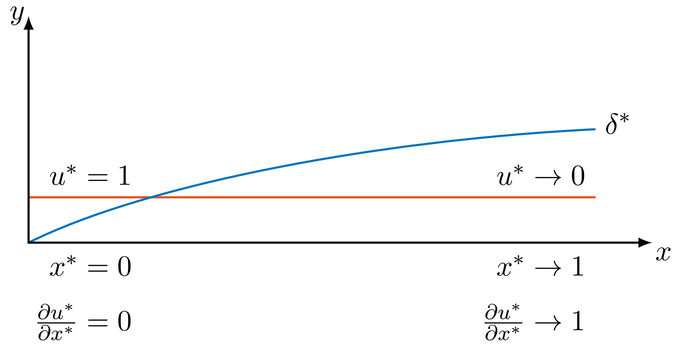

Figure 4: Second-order velocity derivative estimate

From Fig. 4 we see that the non-dimensional velocity gradient in the flow direction changes from $\partial u^*/\partial x^* = 0$ at the start of the boundary layer to $\partial u^*/\partial x^* = 1$ further downstream and thus

$$\frac{\partial^2 u^*}{\partial x^{*2}} \sim 1$$

If we now have a look at the continuity equation (Eqn. 4), we see that

$$\frac{\partial v^*}{\partial y^*} \sim \frac{\partial u^*}{\partial x^*} \sim 1$$

Since $\partial v^*/\partial y^* \sim 1$ and since v^* approaches zero at the wall due to the no-slip condition, v^* must be in the order of δ^* at the outer edge of the boundary layer (at $y^* = \delta^*$) and thus for the greater part of the boundary layer we will have

$$v^* \sim \delta^*$$

The derivatives of v^* that now remains to be analyzed are obtained in the same way as was done for the derivatives of u^*.

Now, if we assume that the boundary layer is very thin compared to its length (a fair assumption for boundary layers) we will get further insight into the importance of the individual terms in the governing equations.

The fact that the boundary layer is assumed to be thin directly gives

$$\delta^* = \frac{\delta}{L} \ll 1$$

The viscous term in the x-component of the momentum equation reads

$$\frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

The first term is in the order of 1 and the second term is in the order of $1/{\delta^*}^2$. With the assumption of thin boundary layers in mind this means that

$$\frac{\partial^2 u^*}{\partial x^{*2}} \ll \frac{\partial^2 u^*}{\partial y^{*2}}$$

In the same way we can see from the y-component of the momentum equation that

$$\frac{\partial^2 v^*}{\partial x^{*2}} \ll \frac{\partial^2 v^*}{\partial y^{*2}}$$

Eqn. 5, now reads

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial y^{*2}} \tag{7}$$

In a boundary layer, friction forces can be assumed to be of the same size as momentum forces. In the left-hand side of Eqn. 7, all terms are in the order of 1 and thus the terms on the right-hand side must also be in the order of 1.

$$\frac{1}{Re}\frac{\partial^2 u^*}{\partial y^{*2}} \sim 1 \Rightarrow Re \sim \frac{1}{\delta^{*2}}$$

Now, let's take a look at the y-component of the momentum equation

$$\overline{u}\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial y^{*2}} \tag{8}$$

With the momentum and friction terms all being in the order of δ^*, we see that the pressure term must be of the same size.

$$\frac{\partial p^*}{\partial y^*} \sim \delta^*$$

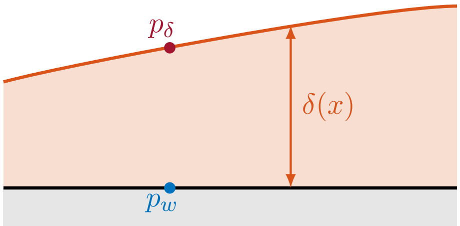

Figure 5: Pressure difference in the boundary layer

This means that if one subtracts the pressure at the wall from the pressure at the edge of the boundary layer one gets a difference that is in the order of ${\delta^*}^2$ (see Fig. 5).

$$p_\delta^* - p_w^* \sim \frac{\partial p^*}{\partial y^*}\delta^* \sim {\delta^*}^2$$

Thus, it is a fair assumption to say that the pressure is independent of the distance from the wall and just a function of x

$$p = p(x)$$

The pressure at the wall, p_w, is in other words equal to the pressure outside of the boundary layer if the boundary layer is thin in relation to its length. This is a result that is used when measuring static pressure.

The analysis of the y-component of the momentum equation gives the result that the pressure is a function of x only and therefore the pressure gradient can be written as a total derivative. We are now done with the analysis of the equations on non-dimensional form and with the new insights, the governing equations for boundary-layer flows on dimensional form reads

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2} \tag{10}$$

The following boundary conditions applies for boundary-layer flows:

$$y = 0 \qquad\qquad u = v = 0$$
$$y \to \delta \qquad\qquad u \to U$$

The pressure gradient in Eqn. 10 can either be obtained from experiments or calculated from the potential flow velocity using the **Bernoulli equation**.

$$p + \frac{1}{2}\rho U^2 = const \Rightarrow \frac{dp}{dx} + \rho U \frac{dU}{dx} = 0$$

and thus

$$-\frac{dp}{dx} = \rho U \frac{dU}{dx} \tag{11}$$

The pressure gradient in the momentum equation may be expressed in terms of a freestream velocity gradient

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = \rho U \frac{dU}{dx} + \mu \frac{\partial^2 u}{\partial y^2} \tag{12}$$

The boundary layer equations can be used for weakly curved surfaces if the coordinate system is set up such that the x-coordinate is aligned with the surface and the y-axis is in the surface normal direction. It is, however, required that the boundary layer thickness, δ, is small in comparison to the surface curvature radius. It is also possible to derive a more generic set of boundary layer equations for three-dimensional unsteady flow.

2.2 Turbulent Boundary Layers

The boundary layer equations for turbulent flows are obtained starting from the **Reynolds-Averaged Navier-Stokes (RANS)** equations. In the same way as for the laminar flow equations, the terms in the equations are compared in terms of size using order of magnitude estimates. For two-dimensional, steady-state flow we get

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0 \tag{13}$$

$$\rho \left(\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} \right) = -\frac{d\overline{p}}{dx} + \mu \frac{\partial^2 \overline{u}}{\partial y^2} - \frac{\partial}{\partial y}\rho\overline{u'v'} \tag{14}$$

A term caused by the turbulent shear stress appears in the equation, which makes it impossible to solve the equations directly. The solution to this closure problem is to either express the turbulence shear stress as a function of averaged flow field properties using a valid hypothesis or to use additional equations for the turbulent properties. In both these cases using experimental data is necessary.

Note that over-lined quantities denote ensemble averages.

2.3 Boundary Layer Thickness

For laminar flows it is possible to estimate the boundary layer thickness directly from the analysis of terms leading up to the boundary layer equations. From section 2.1 we have that

$$Re \sim \frac{1}{{\delta^*}^2} \Rightarrow \delta^* \sim \frac{1}{\sqrt{Re}} \Rightarrow \delta \sim \frac{L}{\sqrt{Re}}$$

The transition from boundary layer to freestream is asymptotic and therefore there are several definitions of boundary layer thickness. Often the boundary layer thickness is defined as the location where the velocity has reached 99% of the freestream velocity, i.e., $u = 0.99U$.

Integral estimates such as the displacement thickness, δ^*, and the momentum thickness, θ, can also be used as measures of the boundary layer thickness.

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$$\theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

In the following these integral estimates will be derived starting from the integral relations.

3 Integral Estimates

3.1 The Von Kármán Integral Relation

An integral momentum relation can be derived by applying the momentum equation for steady-state flow to a control volume aligned with the boundary layer such as the control volume $ABCD$ in Fig. 6.

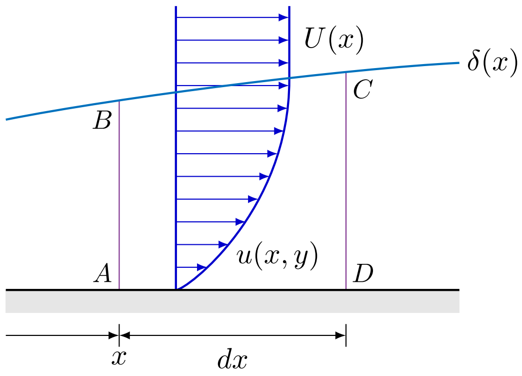

Figure 6: Boundary layer control volume

The extent of the control volume in the z direction is one unit depth.

3.1.1 Mass flow balance

We start by doing a mass flow balance over the control volume ABCD in Fig. 6.

ingoing mass flow through surface AB:

$$\dot{m}_{AB} = \rho \int_0^\delta u dy$$

outgoing mass flow through surface CD:

$$\dot{m}_{CD} = \rho \int_0^\delta u dy + \frac{d}{dx}\left[\rho \int_0^\delta u dy\right] dx = \rho \int_0^\delta u dy + \rho \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

Since surface AD is aligned with the wall and thus there will be no flow going through that surface, there must be a mass flow through surface BC that balance the difference $\dot{m}_{CD} - \dot{m}_{AB}$ and thus

$$\dot{m}_{BC} = \rho \frac{d}{dx}\left[\int_0^\delta u dy\right] dx$$

3.1.2 Momentum balance

The next step is to do a momentum balance over the control volume ABCD in Fig. 6.

ingoing momentum through surface AB:

$$\dot{I}_{AB} = \rho \int_0^\delta u^2 dy$$

outgoing momentum through surface CD:

$$\dot{I}_{CD} = \rho \int_0^\delta u^2 dy + \rho \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] dx$$

ingoing momentum through surface BC:

The massflow through surface BC is known from before and the fluid velocity at the outer edge of the boundary layer equals the freestream velocity and thus

$$\dot{I}_{BC} = U \dot{m}_{BC} = \rho U \frac{d}{dx} \left[\int_0^\delta u dy \right] dx$$

Thus, the net momentum flow out from the control volume is

$$\dot{I}_{CD} - \dot{I}_{AB} - \dot{I}_{BC} = \rho \frac{d}{dx} \left[\int_0^\delta u^2 dy \right] dx - \rho U \frac{d}{dx} \left[\int_0^\delta u dy \right] dx$$

3.1.3 Net force on the control volume

Finally, we need to analyze the forces on the control volume in the x-direction. We will have a friction force along the wall in the negative x-direction. On the other surfaces there will be pressure forces.

$$F_{AD} = -\tau_w dx$$

$$F_{AB} = p\delta$$

$$F_{CD} = -\left(p + \frac{dp}{dx}dx\right)\left(\delta + \frac{d\delta}{dx}dx\right)$$

the pressure force on BC is approximated as

$$F_{BC} \approx \left(p + \frac{1}{2}\frac{dp}{dx}dx\right)\frac{d\delta}{dx}dx$$

The net force excluding second-order terms

$$dF_x = -\tau_w dx - \delta\frac{dp}{dx}dx$$

We can now write out the x-component of the momentum equation for the control volume

$$\rho U \frac{d}{dx}\left[\int_0^\delta u dy\right] - \rho\frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \tau_w + \delta\frac{dp}{dx} \tag{15}$$

This equation is often referred to as the momentum equation for boundary layers or the **von Kármán integral relation**. Eqn. 15 can also be derived from the boundary layer equations Eqn. 10 or Eqn. 14. Eqn. 15 is valid both for laminar and turbulent boundary layers. In the latter, for ensemble-averaged flow variables.

It is possible to rewrite Eqn. 15 and by doing so we will get a relation between the momentum thickness, to be defined, and the wall shear stress.

From Eqn. 11 we have

$$-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx}$$

which can be used to replace the pressure gradient in Eqn. 15

$$\delta\frac{dp}{dx} = -\delta\rho U\frac{dU}{dx} = -\rho U\frac{dU}{dx}\int_0^\delta dy$$

The first term on the left-hand side of Eqn. 15 can be rewritten using the chain rule of differentiation.

$$\rho U \frac{d}{dx}\left[\int_0^\delta u dy\right] = \rho \frac{d}{dx}\left[U \int_0^\delta u dy\right] - \rho \frac{dU}{dx}\int_0^\delta u dy$$

Insert in Eqn. 15 gives

$$\rho \frac{d}{dx}\left[U \int_0^\delta u dy\right] - \rho \frac{dU}{dx}\int_0^\delta u dy - \rho \frac{d}{dx}\left[\int_0^\delta u^2 dy\right] = \tau_w - \rho U \frac{dU}{dx}\int_0^\delta dy$$

Collect terms

$$\rho \frac{d}{dx}\left[U \int_0^\delta u dy - \int_0^\delta u^2 dy\right] + \rho \frac{dU}{dx}\left[U \int_0^\delta dy - \int_0^\delta u dy\right] = \tau_w$$

Dividing by ρ and rewriting the integral terms gives

$$\frac{d}{dx}\int_0^\delta u(U-u)dy + \frac{dU}{dx}\int_0^\delta (U-u)dy = \frac{\tau_w}{\rho} \tag{16}$$

Note that along a flat plate $dU/dx = 0$, which removes the second term on the left-hand side.

3.2 Momentum thickness and displacement thickness

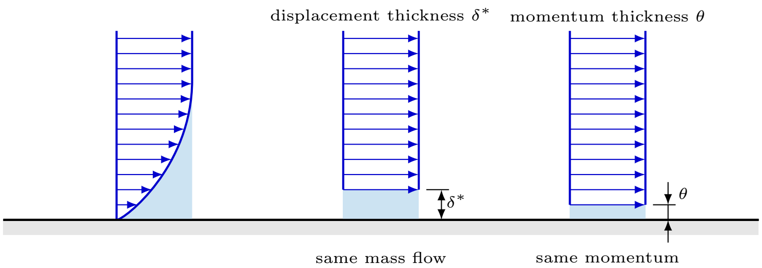

Figure 7: Integral estimates: displacement thickness δ^* and momentum thickness θ

We have now derived the von Kármán integral relation by applying the integral form the continuity and momentum equations to a boundary layer control volume but how can the relation be interpreted physically? As an attempt to describe the physical meaning of the relation, we will have a look at momentum thickness which is one of two integral estimates describing a boundary layer that will be discussed in this section.

The momentum thickness θ is the distance by which a uniform velocity profile must be displaced from the wall in order to give the same deficit of the momentum as the actual velocity profile, see

Fig.7. The deficit of momentum due to the presence of the boundary layer is calculated as follows

$$\int_0^\delta \rho u(U-u)b dy$$

where b is the width of the boundary layer. As described above, the momentum thickness θ is defined as the vertical displacement of a uniform velocity profile that gives the same deficit of momentum as the boundary layer as a result and thus

$$\int_0^\delta \rho u(U-u)b dy = \rho U^2 b\theta$$

The momentum thickness is now obtained as

$$\theta = \int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy \tag{17}$$

Now, if we go back and have a look at the von Kármán integral relation again we can see that it includes the definition of the momentum thickness. If we assume that $dU/dx = 0$, we get

$$\frac{d}{dx} U^2 \underbrace{\int_0^\delta \frac{u}{U}\left(1-\frac{u}{U}\right) dy}_{\theta} = \frac{\tau_w}{\rho}$$

and thus

$$\tau_w = \rho U^2 \frac{d\theta}{dx} \tag{18}$$

Furthermore, we can calculate the total drag due to the presence of the boundary layer as

$$D = \rho b U^2 \theta \tag{19}$$

where, again, b is the width of the boundary layer. From Eqn. 19 it is evident that the momentum thickness is related to the total boundary layer drag.

In the same way as the momentum thickness is calculated from the deficit of momentum due to the presence of the boundary layer, the displacement thickness δ^* is the vertical displacement needed to represent the deficit of mass flow due to the presence of the boundary layer (see

Fig. 7). δ^* is a measure of the boundary layer thickness and gives an estimate of the vertical displacement of streamlines in the outer part of the boundary layer as a consequence of the no-slip condition at the wall. The displacement thickness is calculated as

$$\int_0^\delta \rho(U-u)bdy = \rho U b \delta^* \Rightarrow$$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy \tag{20}$$

The ratio $H = \delta^*/\theta$ is called the shape factor and gives a measure of how prone a boundary layer is to separate. A **Blasius** profile for a laminar flow in a boundary layer with zero pressure gradient has a shape factor of $H = 2.59$. A negative pressure gradient will reduce the shape factor and a positive pressure gradient will increase the value of the shape factor. For a laminar boundary layer, the Reynolds number based on the displacement thickness and the local Reynolds number are related according to

$$Re_{\delta^*} = \frac{\delta^* U}{\nu} \simeq 1.72\sqrt{\frac{xU}{\nu}} = 1.72\sqrt{Re_x}$$

$$\frac{Re_{\delta^*}}{Re_x} \simeq 1.72$$