

Compressible Flow - TME085

Lecture 4

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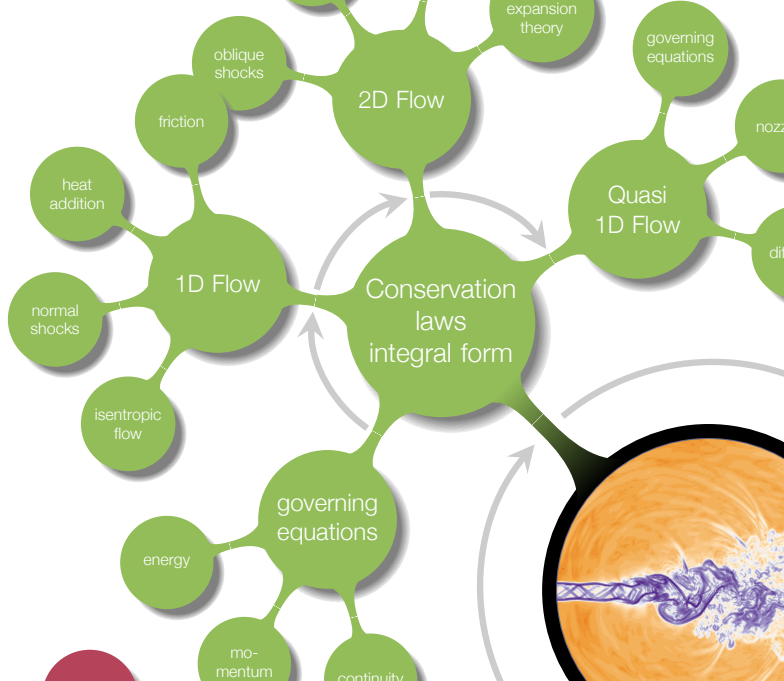


Chapter 3

One-Dimensional Flow



Overview

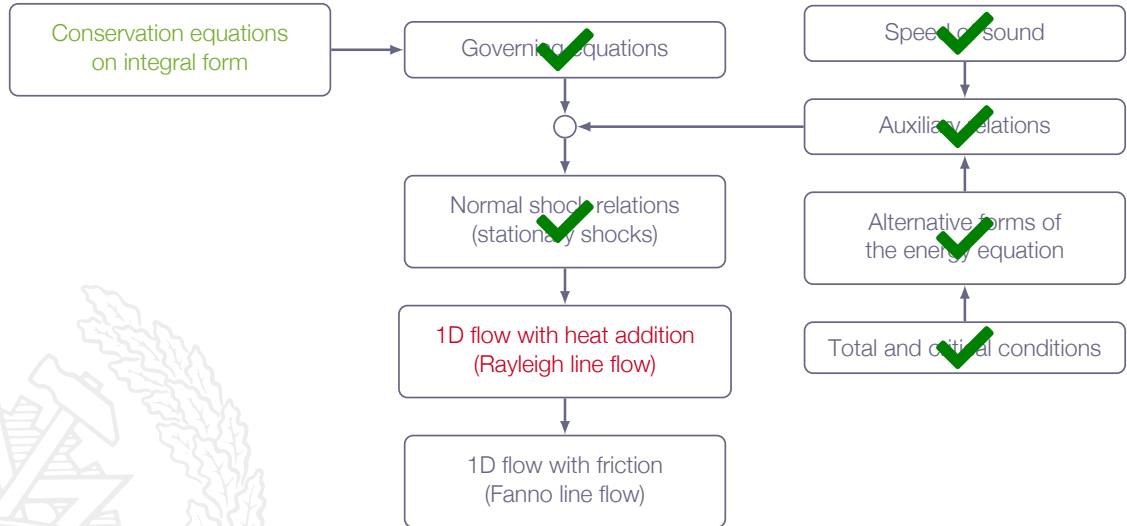


Learning Outcomes

- 4 **Present** at least two different formulations of the governing equations for compressible flows and **explain** what basic conservation principles they are based on
- 5 **Explain** how thermodynamic relations enter into the flow equations
- 6 **Define** the special cases of calorically perfect gas, thermally perfect gas and real gas and **explain** the implication of each of these special cases
- 8 **Derive** (marked) and **apply** (all) of the presented mathematical formulae for classical gas dynamics
 - a 1D isentropic flow*
 - b normal shocks*
 - c 1D flow with heat addition*
 - d 1D flow with friction*

one-dimensional flows - isentropic and non-isentropic

Roadmap - One-dimensional Flow

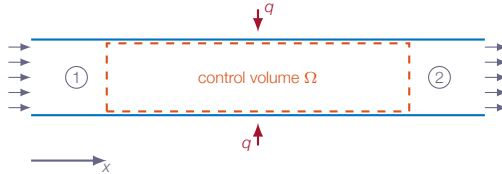


Chapter 3.8

One-Dimensional Flow with Heat Addition



One-Dimensional Flow with Heat Addition



1D pipe flow with heat addition:

1. no friction
2. 1D steady-state $\Rightarrow$ all variables depend on x only
3. q is the amount of heat per unit mass added between 1 and 2
4. analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Heat Addition

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2}u_1^2 + \textcolor{red}{q} = h_2 + \frac{1}{2}u_2^2$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ can be solved analytically

One-Dimensional Flow with Heat Addition

Calorically perfect gas ($h = C_p T$):

$$C_p T_1 + \frac{1}{2} u_1^2 + q = C_p T_2 + \frac{1}{2} u_2^2$$

$$q = \left(C_p T_2 + \frac{1}{2} u_2^2 \right) - \left(C_p T_1 + \frac{1}{2} u_1^2 \right)$$

$$C_p T_o = C_p T + \frac{1}{2} u^2 \Rightarrow$$

$$q = C_p (T_{o2} - T_{o1})$$

i.e. heat addition increases T_o downstream

One-Dimensional Flow with Heat Addition

Momentum equation:

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2$$

$$\left\{ \rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma p}{\rho} M^2 = \gamma p M^2 \right\}$$

$$p_2 - p_1 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2 \Rightarrow$$

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1 . In what way is this relation different than the one for normal shocks – the momentum equation is the same?



Normal Shock Relations

We used the momentum equation to derive the relation for p_2/p_1 . In what way is this relation different than the one for normal shocks – the momentum equation is the same?

Answer: There is no difference. If we would insert $M_2 = f(M_1)$ from the normal shock relations, we would end up with the normal shock relation for p_2/p_1 .

The relation for $M_2 = f(M_1)$ for normal shocks was derived assuming adiabatic flow

One-Dimensional Flow with Heat Addition

Ideal gas law:

$$T = \frac{p}{\rho R} \Rightarrow \frac{T_2}{T_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2} = \frac{\rho_2}{\rho_1} \frac{\rho_1}{\rho_2}$$

Continuity equation:

$$\rho_1 u_1 = \rho_2 u_2 \Rightarrow \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2 \sqrt{\gamma R T_2}}{M_1 \sqrt{\gamma R T_1}} \Rightarrow \frac{\rho_1}{\rho_2} = \frac{M_2}{M_1} \sqrt{\frac{T_2}{T_1}}$$

$$\frac{T_2}{T_1} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right)^2 \left(\frac{M_2}{M_1} \right)^2$$

One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{T_2}{T_1} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right]^2 \left(\frac{M_2}{M_1} \right)^2$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} \right] \left(\frac{M_1}{M_2} \right)^2$$

$$\frac{\rho_2}{\rho_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$



One-Dimensional Flow with Heat Addition

Calorically perfect gas, analytic solution:

$$\frac{p_{o2}}{p_{o1}} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right] \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_2^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2} \right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{T_{o2}}{T_{o1}} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \right] \left(\frac{M_2}{M_1} \right)^2 \left(\frac{1 + \frac{1}{2}(\gamma - 1)M_2^2}{1 + \frac{1}{2}(\gamma - 1)M_1^2} \right)^{\frac{\gamma}{\gamma-1}}$$

One-Dimensional Flow with Heat Addition

Initially subsonic flow ($M < 1$)

the Mach number, M , increases as more heat (per unit mass) is added to the gas
for some limiting heat addition q^* , the flow will eventually become sonic $M = 1$

Initially supersonic flow ($M > 1$)

the Mach number, M , decreases as more heat (per unit mass) is added to the gas
for some limiting heat addition q^* , the flow will eventually become sonic $M = 1$

Note! The (*) condition in this context is not the same as the "critical" condition discussed for isentropic flow

One-Dimensional Flow with Heat Addition

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}$$

Calculate the ratio between the pressure at a specific location in the flow p and the pressure at sonic conditions p^*

$$p_1 = p, M_1 = M, p_2 = p^*, \text{ and } M_2 = 1$$

$$\frac{p^*}{p} = \frac{1 + \gamma M^2}{1 + \gamma}$$

One-Dimensional Flow with Heat Addition

$$\frac{T}{T^*} = \left[\frac{1 + \gamma}{1 + \gamma M^2} \right]^2 M^2$$

$$\frac{p_o}{p_o^*} = \left[\frac{1 + \gamma}{1 + \gamma M^2} \right] \left(\frac{2 + (\gamma - 1)M^2}{(\gamma + 1)} \right)^{\frac{\gamma}{\gamma - 1}}$$

$$\frac{\rho}{\rho^*} = \left[\frac{1 + \gamma M^2}{1 + \gamma} \right] \left(\frac{1}{M^2} \right)$$

$$\frac{T_o}{T_o^*} = \frac{(\gamma + 1)M^2}{(1 + \gamma M^2)^2} (2 + (\gamma - 1)M^2)$$

$$\frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2}$$

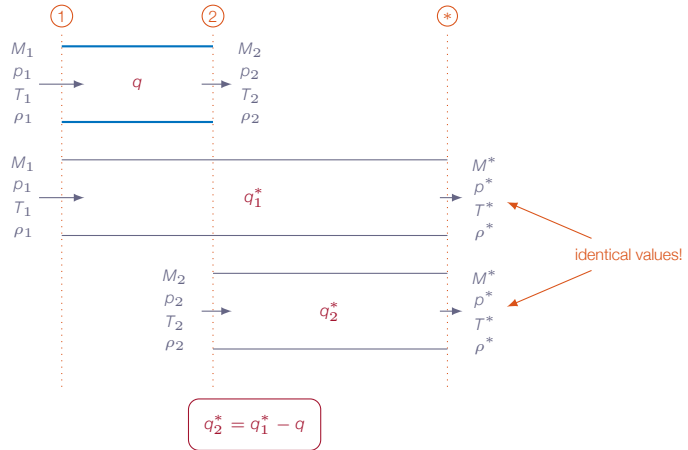
One-Dimensional Flow with Heat Addition

Amount of heat per unit mass needed to choke the flow:

$$q^* = C_p(T_o^* - T_o) = C_p T_o \left(\frac{T_o^*}{T_o} - 1 \right)$$



One-Dimensional Flow with Heat Addition



Note! for a given flow, the starred quantities are constant values

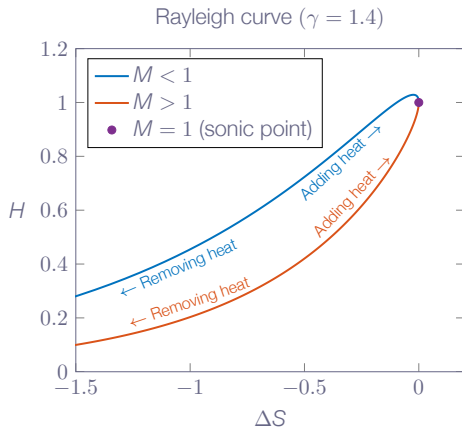
One-Dimensional Flow with Heat Addition



Lord Rayleigh 1842-1919
Nobel prize in physics 1904

Note! it is theoretically possible to heat an initially subsonic flow to reach sonic conditions and then continue to accelerate the flow by cooling

$$\Delta S = \frac{\Delta s}{C_p} = \ln \left[M^2 \left(\frac{\gamma + 1}{1 + \gamma M^2} \right)^{\frac{\gamma + 1}{\gamma}} \right]$$
$$H = \frac{h}{h^*} = \frac{C_p T}{C_p T^*} = \frac{T}{T^*} = \left[\frac{(\gamma + 1)M}{1 + \gamma M^2} \right]^2$$



One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...



One-Dimensional Flow with Heat Addition

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Removing heat seems to reduce the entropy. Isn't that a violation of the second law of thermodynamics?!

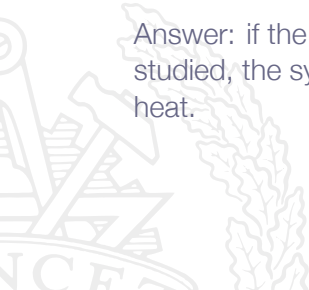


One-Dimensional Flow with Heat Addition

And now, the million-dollar question ...

Removing heat seems to reduce the entropy. Isn't that a violation of the second law of thermodynamics?!

Answer: if the heat source or sink would have been included in the system studied, the system entropy would increase both when adding and removing heat.



One-Dimensional Flow with Heat Addition

$M < 1$: Adding heat will

increase M
decrease p
increase T_o
decrease p_o
increase s
increase u
decrease ρ

$M > 1$: Adding heat will

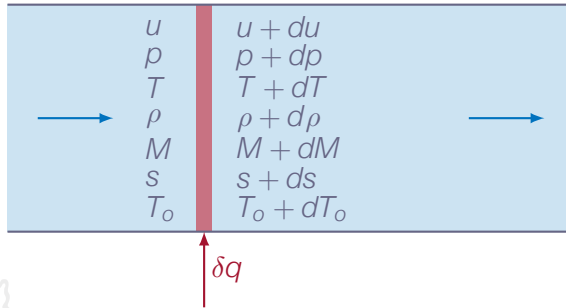
decrease M
increase p
increase T_o
decrease p_o
increase s
decrease u
increase ρ

Note! the flow is not isentropic, there will always be losses

The Rayleigh-flow Process

Unlike the normal shock, Rayleigh flow has **continuous** solutions

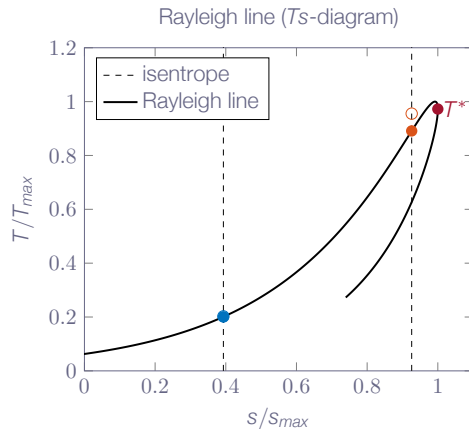
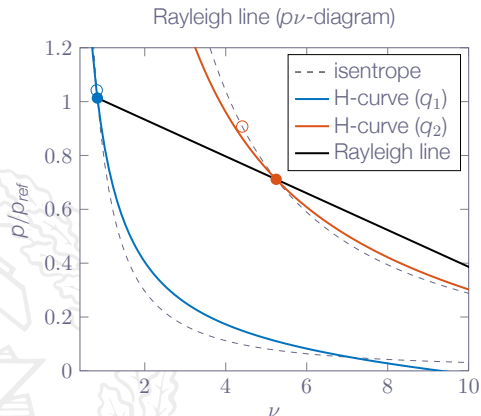
A small addition of heat δq will change flow properties slightly



The Rayleigh-flow Process - Subsonic Heat Addition

Note!

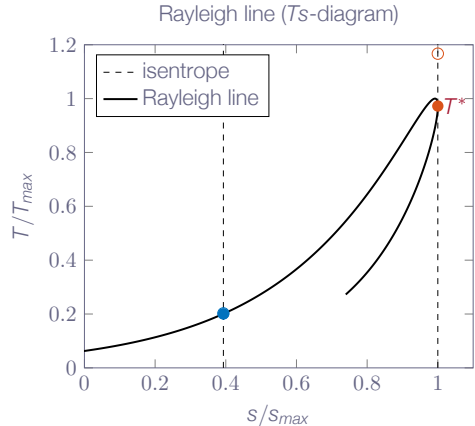
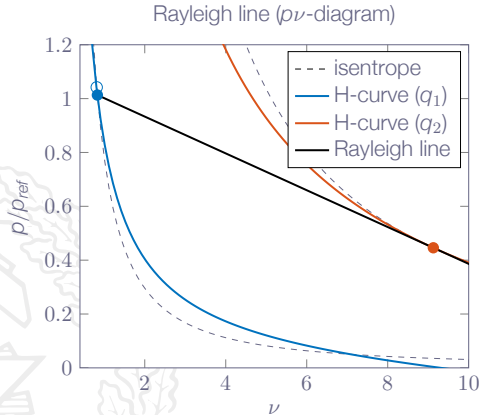
Heat addition moves the H-curve in the direction of increasing pressure and increasing specific volume



The Rayleigh-flow Process - Choked Subsonic Flow

Note!

When $q = q^*$, the H-curve is tangent to the Rayleigh line (thermal choking)
Further heat addition will move the H-curve away from the Rayleigh line

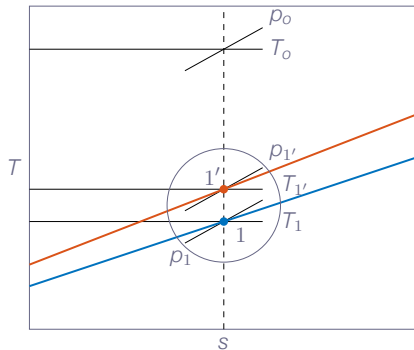
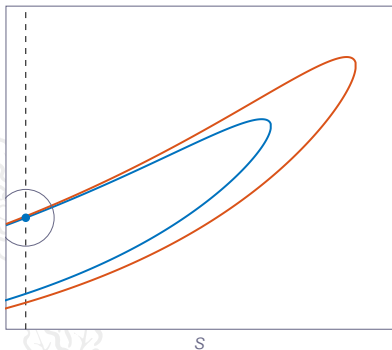


The Rayleigh-flow Process - Choked Subsonic Flow

Note!

If it is added such that $q > q^*$, the inlet static flow properties will change (new mass-flow) such that the new q^* is equal to the added heat q

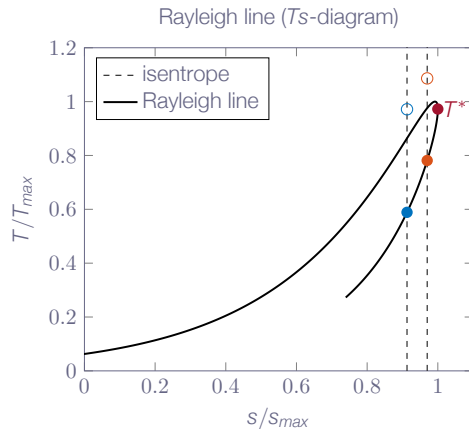
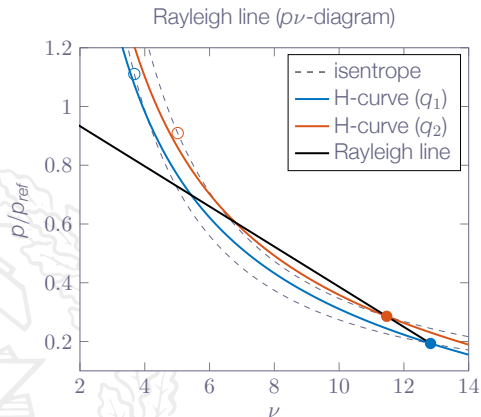
Total flow properties at the inlet remains the same (only work or heat addition can change the total flow properties)



The Rayleigh-flow Process - Supersonic Heat Addition

Note!

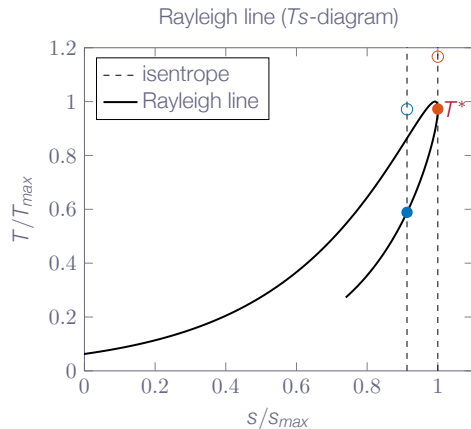
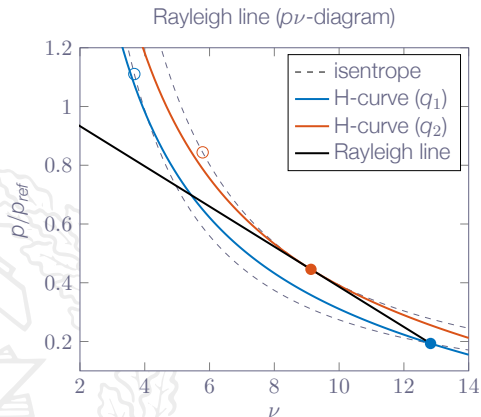
A supersonic flow is in general closer to thermal choking than a subsonic flow due to the high energy level (and thereby high T_o)



The Rayleigh-flow Process - Choked Supersonic Flow

Note!

When heat is added to a thermally choked supersonic flow, a shock will be generated at the exit of the pipe



The Rayleigh-flow Process - Choked Supersonic Flow

The shock generated at the exit will be infinitely weak ($M = 1$)

As the shock does not affect T_o , T^* , p^* etc, it does not affect the thermal choking condition (remember: T^* and p^* are **not the critical conditions**)

The heat process and the normal shock process operates along the **same line** in $p\nu$ -space

The shock will travel upstream through the pipe

If the supersonic flow is generated in a convergent-divergent nozzle, the shock will propagate upstream in the nozzle until the resulting pipe inlet condition allows for the heat to be added with thermal choking at the pipe exit

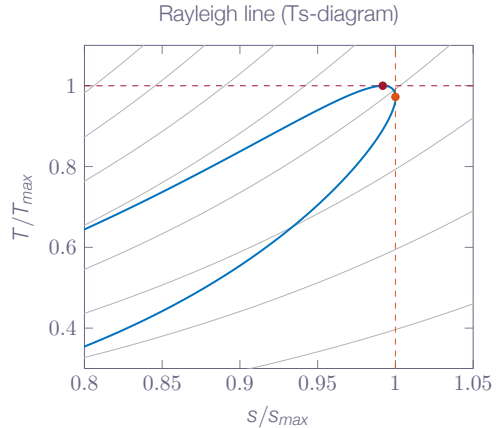
The Rayleigh-flow Process - Maximum Temperature

It can be showed that $\frac{dT}{ds} = \frac{1 - \gamma M^2}{1 - M^2} \frac{T}{C_p}$

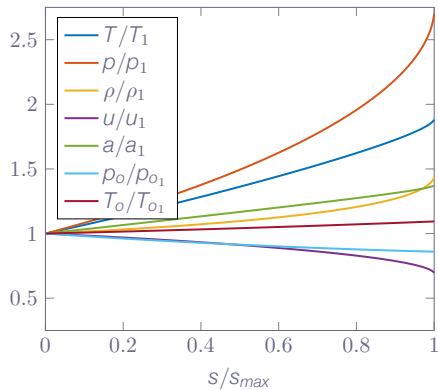
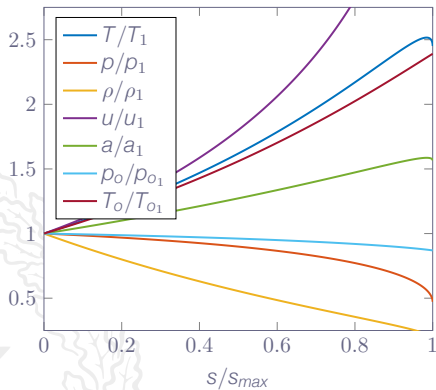
$$\frac{dT}{ds} = 0 \Rightarrow M = \sqrt{\frac{1}{\gamma}}$$

we will have the maximum temperature for a subsonic Mach number

$$M = 1.0 \Rightarrow \frac{dT}{ds} = \infty$$

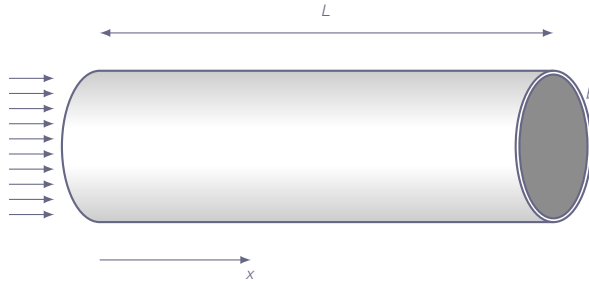


Rayleigh Flow Trends



One-Dimensional Flow with Heat Addition

Relation between added heat per unit mass (q) and heat per unit surface area and unit time ($\dot{q}_{wall}$)



Pipe with arbitrary cross section (constant in x):

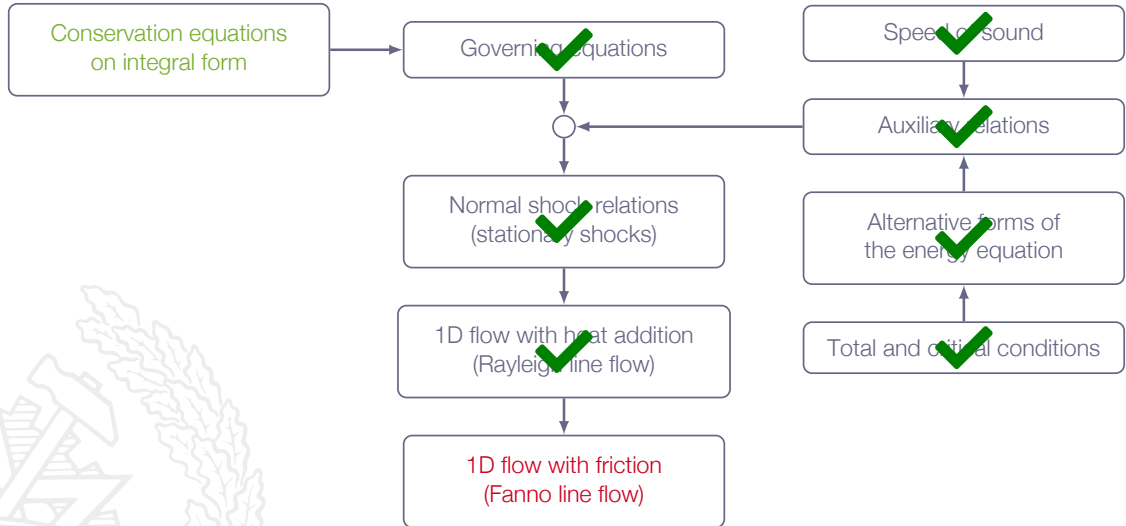
mass flow through pipe $\dot{m}$

axial length of pipe L

circumference of pipe $b = 2\pi r$

$$q = \frac{Lb\dot{q}_{wall}}{\dot{m}}$$

Roadmap - One-dimensional Flow



Chapter 3.9

One-Dimensional Flow with Friction

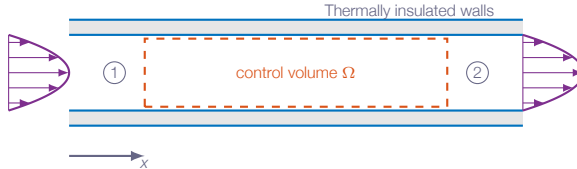


One-Dimensional Flow with Friction

inviscid flow with friction?!



One-Dimensional Flow with Friction



1D pipe flow with friction:

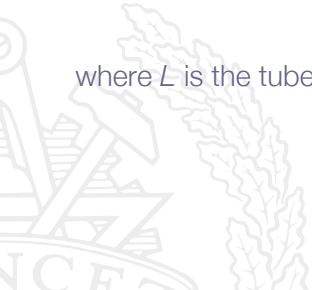
1. adiabatic ($q = 0$)
2. cross section area A is constant
3. average all variables in each cross-section $\Rightarrow$ only x -dependence
4. analyze by setting up a control volume between station 1 and 2

One-Dimensional Flow with Friction

Wall-friction contribution in momentum equation

$$\oiint_{\partial\Omega} \tau_w dS = b \int_0^L \tau_w dx$$

where L is the tube length and b is the circumference



One-Dimensional Flow with Friction

$$\rho_1 u_1 = \rho_2 u_2$$

$$\rho_1 u_1^2 + p_1 - \frac{4}{D} \int_0^L \tau_w dx = \rho_2 u_2^2 + p_2$$

$$h_1 + \frac{1}{2} u_1^2 = h_2 + \frac{1}{2} u_2^2$$



One-Dimensional Flow with Friction

τ_w varies with the distance x and thus complicating the integration

Solution: let L shrink to dx and we end up with relations on differential form

$$d(\rho u^2 + p) = -\frac{4}{D}\tau_w dx \Leftrightarrow \frac{d}{dx}(\rho u^2 + p) = -\frac{4}{D}\tau_w$$



One-Dimensional Flow with Friction

From the continuity equation we get

$$\rho_1 u_1 = \rho_2 u_2 = \text{const} \Rightarrow \frac{d}{dx}(\rho u) = 0$$

Writing out all terms in the momentum equation gives

$$\frac{d}{dx}(\rho u^2 + p) = \rho u \frac{du}{dx} + u \underbrace{\frac{d}{dx}(\rho u)}_{=0} + \frac{dp}{dx} = -\frac{4}{D}\tau_w \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{4}{D}\tau_w$$

Common approximation for τ_w :

$$\tau_w = f \frac{1}{2} \rho u^2 \Rightarrow \rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D} \rho u^2 f$$

One-Dimensional Flow with Friction

Energy conservation:

$$h_{o1} = h_{o2} \Rightarrow \frac{d}{dx}h_o = 0$$



One-Dimensional Flow with Friction

Summary:

$$\frac{d}{dx}(\rho u) = 0$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = -\frac{2}{D} \rho u^2 f$$

$$\frac{d}{dx} h_o = 0$$

Valid for all gases!

General gas $\Rightarrow$ Numerical solution necessary

Calorically perfect gas $\Rightarrow$ Can be solved analytically (for constant f)

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\int_{x_1}^{x_2} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma + 1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} \right) \right]_{M_1}^{M_2}$$



One-Dimensional Flow with Friction

Calorically perfect gas and adiabatic flow:

$$\frac{T_2}{T_1} = \frac{T_2}{T_{o2}} \frac{T_{o2}}{T_{o1}} \frac{T_{o1}}{T_1} = \{T_o = \text{const}\} = \frac{T_2}{T_o} \frac{T_o}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

Continuity:

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{a_1 M_1}{a_2 M_2} = \left\{ a = \sqrt{\gamma R T} \right\} = \sqrt{\frac{T_1}{T_2}} \left(\frac{M_1}{M_2} \right)$$

Perfect gas:

$$\frac{p_2}{p_1} = \{p = \rho R T\} = \frac{\rho_2 T_2}{\rho_1 T_1}$$

Total pressure:

$$\frac{p_{o2}}{p_{o1}} = \frac{p_{o2}}{p_2} \frac{p_2}{p_1} \frac{p_1}{p_{o1}}$$

One-Dimensional Flow with Friction

Calorically perfect gas:

$$\frac{T_2}{T_1} = \frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2}$$

$$\frac{\rho_2}{\rho_1} = \frac{M_1}{M_2} \left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2} \right]^{-1/2}$$

$$\frac{p_2}{p_1} = \frac{M_1}{M_2} \left[\frac{2 + (\gamma - 1)M_1^2}{2 + (\gamma - 1)M_2^2} \right]^{1/2}$$

$$\frac{p_{o2}}{p_{o1}} = \frac{M_1}{M_2} \left[\frac{2 + (\gamma - 1)M_2^2}{2 + (\gamma - 1)M_1^2} \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

One-Dimensional Flow with Friction

Initially subsonic flow ($M_1 < 1$)

M_2 will increase as L increases

for a critical length L^* , the flow at point 2 will reach sonic conditions, i.e. $M_2 = 1$

Initially supersonic flow ($M_1 > 1$)

M_2 will decrease as L increases

for a critical length L^* , the flow at point 2 will reach sonic conditions, i.e. $M_2 = 1$

Note! The (*) condition in this context is not the same as the "critical" condition discussed for isentropic flow

One-Dimensional Flow with Friction

$$\frac{T}{T^*} = \frac{(\gamma + 1)}{2 + (\gamma - 1)M^2}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{1/2}$$

$$\frac{p}{p^*} = \frac{1}{M} \left[\frac{\gamma + 1}{2 + (\gamma - 1)M^2} \right]^{1/2}$$

$$\frac{p_o}{p_o^*} = \frac{1}{M} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

see Table A.4

One-Dimensional Flow with Friction

and

$$\int_0^{L^*} \frac{4f}{D} dx = \left[-\frac{1}{\gamma M^2} - \frac{\gamma + 1}{2\gamma} \ln \left(\frac{M^2}{1 + \frac{\gamma - 1}{2} M^2} \right) \right]_M^1$$

where L^* is the tube length needed to change current state to sonic conditions

Let $\bar{f}$ be the average friction coefficient over the length $L^* \Rightarrow$

$$\frac{4\bar{f}L^*}{D} = \frac{1 - M^2}{\gamma M^2} + \frac{\gamma + 1}{2\gamma} \ln \left(\frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2} \right)$$

Turbulent pipe flow $\rightarrow \bar{f} \sim 0.005$ ($Re > 10^5$, roughness $\sim 0.001D$)

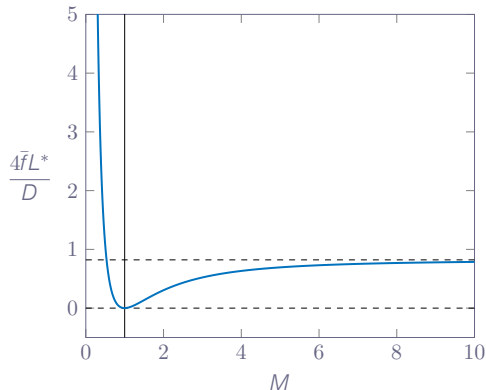
One-Dimensional Flow with Friction - Choking Length

Note!

Supersonic flow is much more prone to choke than subsonic flow

There is an upper limit for supersonic choking length L^*

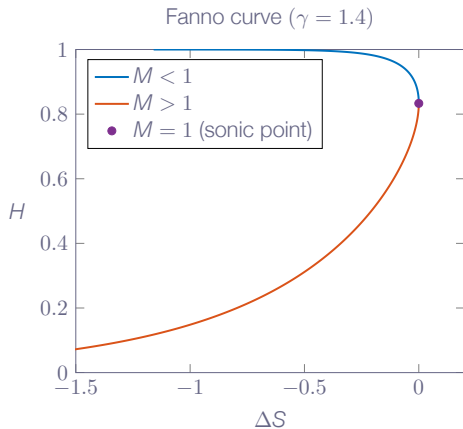
$$\left. \frac{4\bar{f}L^*}{D}(M_1) \right|_{M_1 \rightarrow \infty} = \frac{1}{\gamma} + \left(\frac{\gamma + 1}{2\gamma} \right) \ln \left(\frac{\gamma + 1}{\gamma - 1} \right)$$



One-Dimensional Flow with Friction

$$H = \frac{h}{h_o} = \frac{C_p T}{C_p T_o} = \frac{T}{T_o} = \left[1 + \frac{\gamma - 1}{2} M^2 \right]^{-1}$$

$$\Delta S = \frac{\Delta s}{C_p} = \ln \left[\left(\frac{1}{H} - 1 \right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{2}{\gamma-1} \right)^{\frac{\gamma-1}{2\gamma}} \left(\frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{2\gamma}} (H)^{\frac{\gamma+1}{2\gamma}} \right]$$



One-Dimensional Flow with Friction

$M < 1$: Friction will

increase M
decrease p
decrease T
decrease p_o
increase s
increase u
decrease ρ

$M > 1$: Friction will

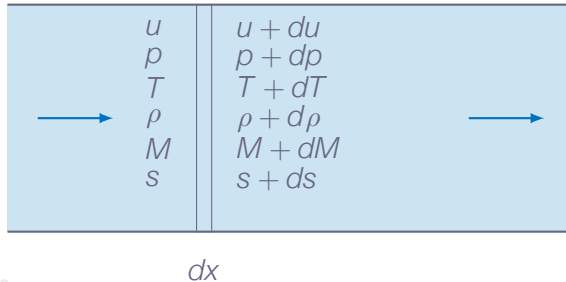
decrease M
increase p
increase T
decrease p_o
increase s
decrease u
increase ρ

Note! the flow is not isentropic, there will always be losses

The Fanno-flow Process

Just like the Rayleigh flow, Fanno flow has **continuous** solutions

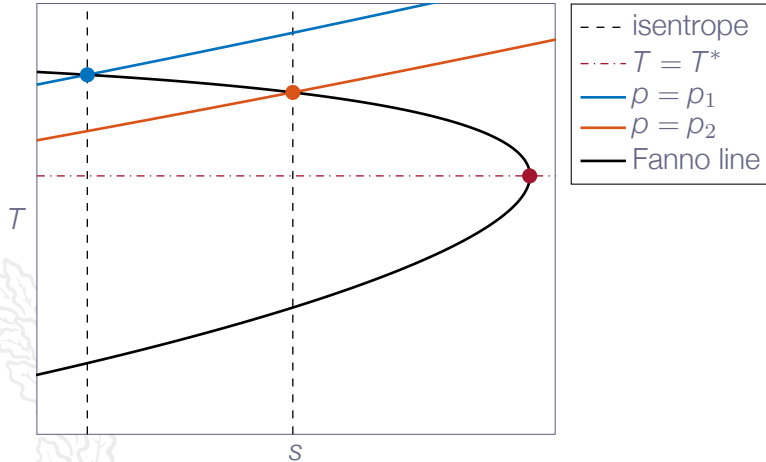
A small pipe section with length dx will change flow properties slightly



The Fanno-flow Process - Subsonic Flow

Note!

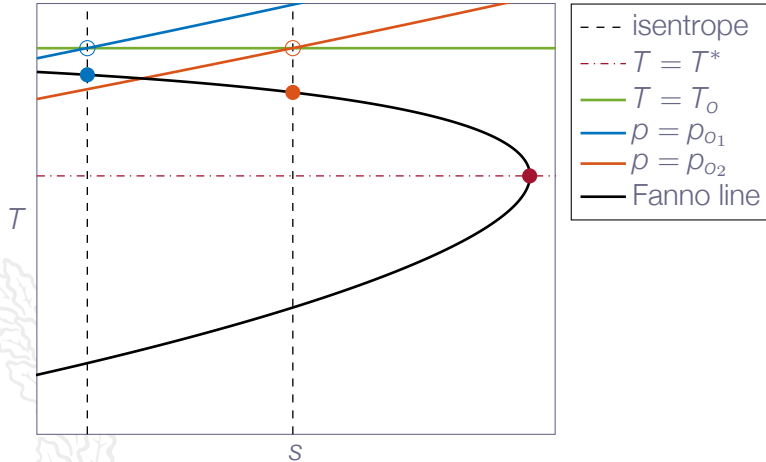
Pressure and temperature decreases when friction is added to a subsonic flow



The Fanno-flow Process - Subsonic Flow

Note!

The Fanno flow process is adiabatic $\Rightarrow T_o$ is constant $\Rightarrow p_o$ increases

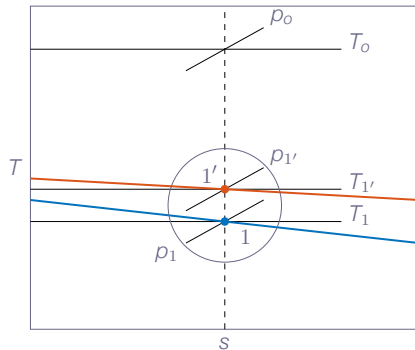
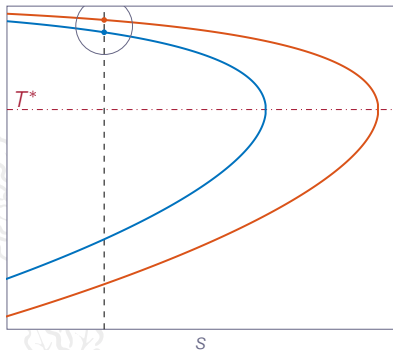


The Fanno-flow Process - Choked Subsonic Flow

Note!

If the pipe length is increased such that $L > L^*$, the inlet static flow properties will change (new massflow) such that the new L^* is equal to the pipe length

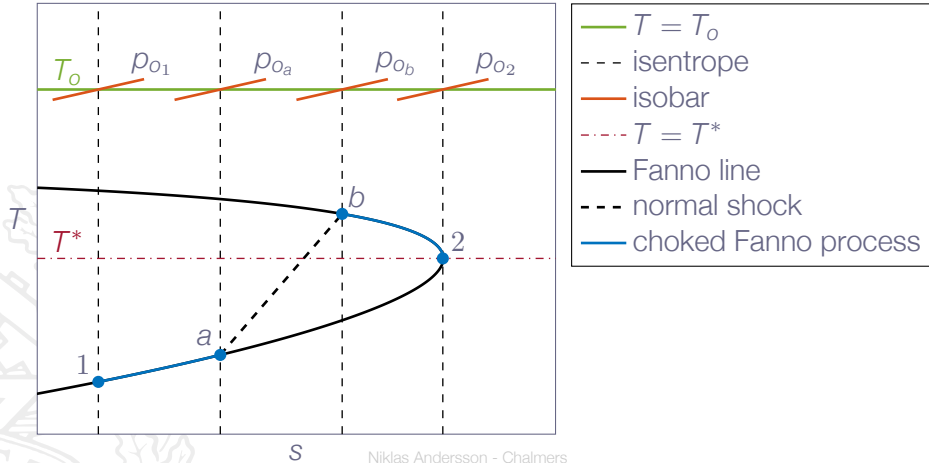
Total flow properties at the inlet remains the same (only work or heat addition can change the total flow properties)



The Fanno-flow Process - Choked Supersonic

Note!

Choked supersonic flow will lead to the formation of a shock inside the pipe (shock location depends on flow conditions)



The Fanno-flow Process - Choked Supersonic

Why does the normal shock change the choking condition for Fanno flow but not for Rayleigh flow?

As for Rayleigh flow, T_o , T^* , p^* , etc are not affected by the shock

The **momentum equation is not the same** as for normal shocks $\Rightarrow$ the Fanno-flow process does not operate along the same line as the normal-shock process in $p\nu$ -space

The Fanno-flow Process - Choked Supersonic

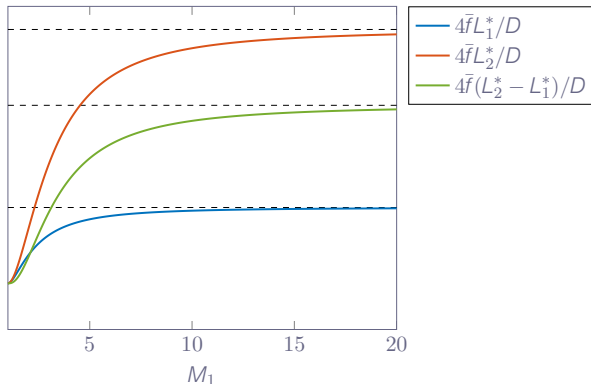
Note!

An internal shock will always increase the choking length L^*

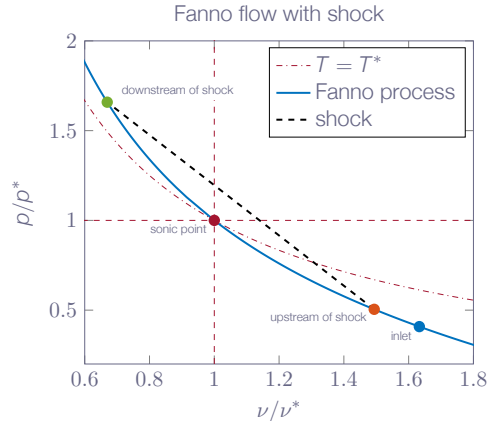
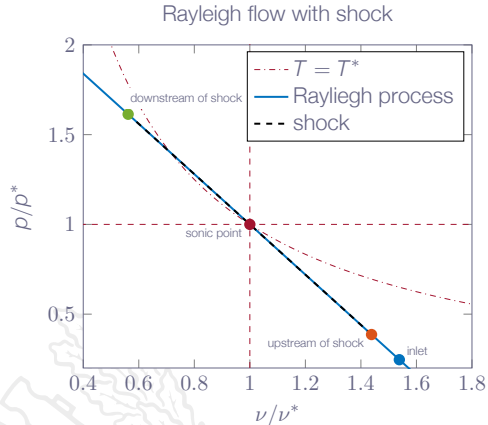
$$L_1^* = f(M_1)$$

$$\left. \begin{array}{l} L_2^* = f(M_2) \\ M_2 = f(M_1) \end{array} \right\} \Rightarrow L_2^* = f(M_1)$$

$$L_2^* - L_1^* = f(M_1)$$



Friction Choking vs Thermal Choking

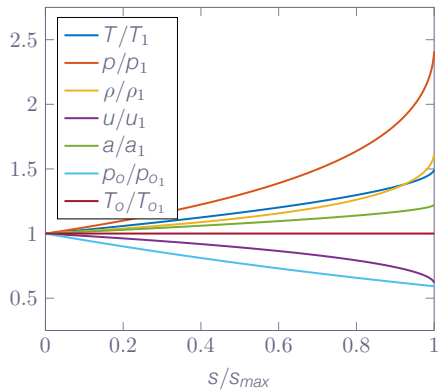
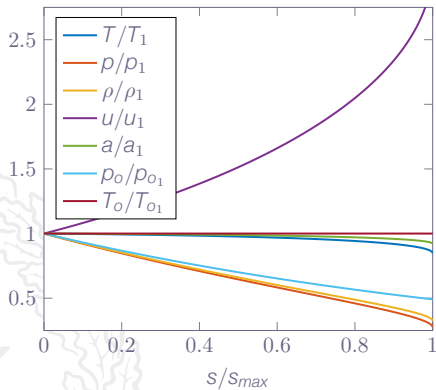


Rayleigh flow: a shock does not affect T_o^* or $T_o \Rightarrow q^*$ will not change over the shock

Fanno flow: L^* changes discontinuously over the shock $\Rightarrow$

L^* will always increase over a shock $\Rightarrow$ possible to extend pipe for supersonic flow

Fanno-flow Trends



Roadmap - One-dimensional Flow

