



Compressible Flow TME085

Quasi-One-Dimensional Flow

The Area-Mach-Number Relation

Division of Fluid Dynamics

Department of Mechanics and Maritime Sciences

Chalmers University of Technology

Governing Equations

The governing equations for quasi-one-dimensional form are the continuity, momentum, and energy equations on differential form where a term accounting for the variation in cross-section area is added in the momentum equation.

$$d(\rho u A) = 0 \quad (1)$$

$$dp = -\rho u du \quad (2)$$

$$dh + u du = 0 \quad (3)$$

The Area-Velocity Relation

Starting point - the continuity equation (Eqn. 1):

$$d(\rho u A) = 0 \Rightarrow \rho u A = \text{const}$$

This applies everywhere in the nozzle and therefore the sonic conditions can be used as a reference

$$\rho u A = \rho^* u^* A^* = \{u^* = a^*\} = \rho^* a^* A^*$$

divide by $\rho u A^*$ gives

$$\frac{\rho^*}{\rho} \frac{a^*}{u} = \frac{A}{A^*}$$

$a^*/u = 1/M^*$ but ρ^*/ρ is unknown

$$\frac{\rho^*}{\rho} = \frac{\rho^*}{\rho_o} \frac{\rho_o}{\rho}$$

and thus

$$\frac{\rho^*}{\rho_o} \frac{\rho_o}{\rho} \frac{1}{M^*} = \frac{A}{A^*} \quad (4)$$

Using the isentropic relations, we get

$$\frac{\rho^*}{\rho_o} = \frac{1}{\left[\frac{1}{2}(\gamma - 1) \right]^{1/(\gamma-1)}} \quad (5)$$

$$\frac{\rho_o}{\rho} = \left[1 + \frac{1}{2}(\gamma + 1)M^2 \right]^{1/(\gamma-1)} \quad (6)$$

Eqns. 5 and 6 in Eqn. 4 gives

$$\frac{A}{A^*} = \frac{1}{M^*} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{1/(\gamma-1)} \quad (7)$$

What remains now is to replace M^*

$$M^{*2} = \frac{u^2}{a^{*2}} = \frac{u^2}{a^2} \frac{a^2}{a^{*2}} = \frac{u^2}{a^2} \frac{a^2}{a_o^2} \frac{a_o^2}{a^{*2}} = M^2 \frac{a^2}{a_o^2} \frac{a_o^2}{a^{*2}} \quad (8)$$

For a calorically perfect gas $a = \sqrt{\gamma RT}$, which gives

$$\frac{a^2}{a_o^2} = \frac{T}{T_o} = \left[1 + \frac{1}{2}(\gamma - 1)M^2 \right]^{-1} \quad (9)$$

$$\frac{a_o^2}{a^{*2}} = \frac{T_o}{T^*} = \frac{1}{2}(\gamma + 1) \quad (10)$$

Eqns. 9 and 10 in Eqn. 8 gives

$$M^{*2} = \frac{(\gamma + 1)M^2}{2 + (\gamma - 1)M^2} \quad (11)$$

Now, rewrite Eqn. 7 as

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^{*2}} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{2/(\gamma-1)} \quad (12)$$

and insert M^{*2} from Eqn. 11

$$\begin{aligned} \left(\frac{A}{A^*}\right)^2 &= \frac{2 + (\gamma - 1)M^2}{(\gamma + 1)M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{2/(\gamma-1)} \Rightarrow \\ \left(\frac{A}{A^*}\right)^2 &= \frac{1}{M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{1+2/(\gamma-1)} \Rightarrow \\ \left(\frac{A}{A^*}\right)^2 &= \frac{1}{M^2} \left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1} \right]^{(\gamma+1)/(\gamma-1)} \end{aligned} \quad (13)$$

which is the area-Mach-number relation.

For a nozzle flow, the area-Mach-number relation gives the Mach number, M , at any location inside the nozzle as a function of the ratio between the local cross-section area, A , and the throat area at choked conditions, A^* .

$$M = f\left(\frac{A}{A^*}\right)$$

Due to the assumptions made in the derivation, the area-Mach-number relation is only valid for isentropic flows of calorically perfect gases. This means that it cannot be used throughout the divergent part of a convergent-divergent nozzle in case there is a shock within the nozzle. It can, however, be used both upstream and downstream of the shock. Note that A^* will change over the shock.

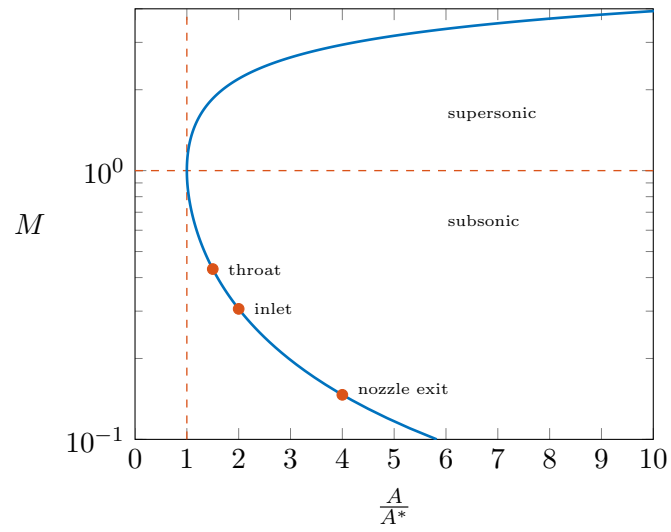


Figure 1: Area-Mach-number relation - subsonic nozzle flow

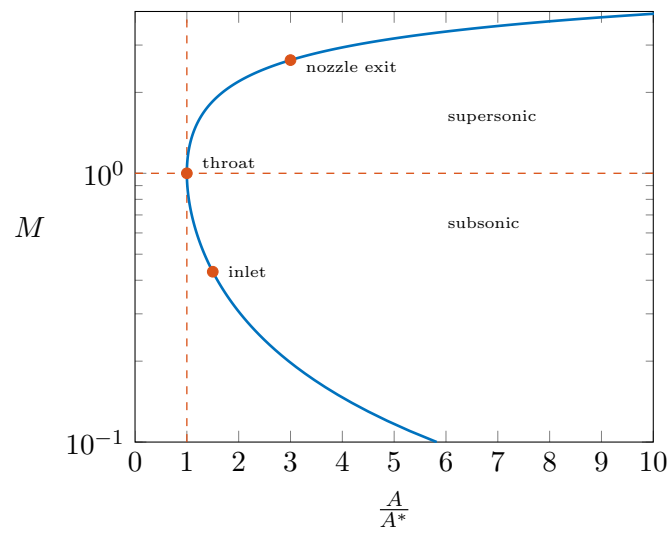


Figure 2: Area-Mach-number relation - supersonic nozzle flow