



Compressible Flow TME085

One-Dimensional Flow with Friction

Details on the derivation of the momentum equation for Fanno flows

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From chapter 3.9 we have the following expression for the momentum equation for one-dimensional flow with friction (equation (3.95))

$$dp + \rho u du = -\frac{1}{2} \rho u^2 \frac{4f dx}{D} \quad (3.95)$$

For cases dealing with calorically perfect gas, (3.95) can be recast completely in terms of Mach number using the following relations

speed of sound: $a^2 = \gamma p / \rho$

the definition of Mach number: $M^2 = u^2 / a^2$

the ideal gas law for thermally perfect gas: $p = \rho R T$

the continuity equation: $\rho u = \text{const}$

energy equation: $c_p T + u^2 / 2 = \text{const}$

1 Continuity equation

We start with the continuity equation which for one-dimensional steady flows reads

$$\rho u = \text{const} \quad (1)$$

Differentiating (1) gives

$$d(\rho u) = 0. \Leftrightarrow \rho du + u d\rho = 0. \quad (2)$$

If $u \neq 0$. we can divide by ρu which gives us

$$\frac{du}{u} + \frac{d\rho}{\rho} = 0. \quad (3)$$

Now, if we divide and multiply the first term in (3) by $2u$ and use the chain rule for derivatives we get

$$\frac{d(u^2)}{2u^2} + \frac{d\rho}{\rho} = 0. \quad (4)$$

2 Energy equation

For an adiabatic one-dimensional flow we have that

$$c_p T + \frac{u^2}{2} = \text{const} \quad (5)$$

If we differentiate (5) we get

$$c_p dT + \frac{1}{2} d(u^2) = 0. \quad (6)$$

We replace c_p with $\gamma R/(\gamma - 1)$ and multiply and divide the first term with T which gives us

$$\frac{\gamma R T}{(\gamma - 1)} \frac{dT}{T} + \frac{1}{2} d(u^2) = 0. \quad (7)$$

Now, divide by $\gamma R T/(\gamma - 1)$ and multiply and divide the second term by u^2 gives

$$\frac{dT}{T} + \frac{(\gamma - 1)}{2} M^2 \frac{d(u^2)}{u^2} = 0. \quad (8)$$

We want to remove the dT/T -term in (8). From the definition of Mach number we have that

$$a^2 M^2 = u^2 \quad (9)$$

which we can rewrite using the expression for speed of sound ($a^2 = \gamma R T$) according to

$$\gamma R T M^2 = u^2 \quad (10)$$

Differentiating (10) gives us

$$\gamma R M^2 dT + \gamma R T d(M^2) = d(u^2) \quad (11)$$

Now, if we divide (11) by $\gamma RT M^2$ and use $a^2 = \gamma RT$ and $a^2 M^2 = u^2$ we get

$$\frac{dT}{T} + \frac{d(M^2)}{M^2} = \frac{d(u^2)}{u^2} \quad (12)$$

Equation (12) may now be used to replace the dT/T -term in equation (8)

$$-\frac{d(M^2)}{M^2} + \frac{d(u^2)}{u^2} + \frac{(\gamma - 1)}{2} M^2 \frac{d(u^2)}{u^2} = 0. \quad (13)$$

which can be rewritten according to

$$\frac{d(u^2)}{u^2} = \left[1 + \frac{(\gamma - 1)}{2} M^2 \right]^{-1} \frac{d(M^2)}{M^2} \quad (14)$$

Using the chain rule for derivatives, the last term may be rewritten according to

$$\frac{d(M^2)}{M^2} = 2M \frac{dM}{M^2} = 2 \frac{dM}{M}$$

which gives

$$\frac{d(u^2)}{u^2} = 2 \left[1 + \frac{(\gamma - 1)}{2} M^2 \right]^{-1} \frac{dM}{M} \quad (15)$$

3 The ideal gas law

For a perfect gas the ideal gas law reads

$$p = \rho RT \quad (16)$$

Differentiating (16) gives:

$$dp = \rho R dT + RT d\rho \quad (17)$$

If $p \neq 0$., we can divide (20) by p which gives

$$\frac{dp}{p} = \frac{dT}{T} + \frac{d\rho}{\rho} \quad (18)$$

which can be rearranged according to

$$\left[\frac{dp}{p} - \frac{d\rho}{\rho} \right] = \frac{dT}{T} \quad (19)$$

Now, inserting dT/T from equation (8) gives

$$\left[\frac{dp}{p} - \frac{d\rho}{\rho} \right] + \frac{(\gamma - 1)}{2} M^2 \frac{d(u^2)}{u^2} = 0. \quad (20)$$

The $d\rho/\rho$ -term can be replaced using equation (4)

$$\frac{dp}{p} + \frac{d(u^2)}{2u^2} + \frac{(\gamma - 1)}{2} M^2 \frac{d(u^2)}{u^2} = 0. \quad (21)$$

Collect terms and rewrite gives

$$\frac{dp}{p} + \left[\frac{1 + (\gamma - 1)M^2}{2} \right] \frac{d(u^2)}{u^2} = 0. \quad (22)$$

4 Momentum equation

By combining the above derived relations and the momentum equation on the form given by (3.95), we can get an expression where the friction force is a function of Mach number only

For convenience equation (3.95) is written again here

$$dp + \rho u du = -\frac{1}{2} \rho u^2 \frac{4f dx}{D} \quad (3.95)$$

if $u \neq 0$, we can divide by $0.5\rho u^2$ which gives

$$2\frac{dp}{\rho u^2} + 2\frac{\rho u du}{\rho u^2} = -\frac{4f dx}{D} \quad (23)$$

using $M^2 = u^2/a^2$, $a^2 = \gamma p/\rho$ and the chain rule in (23) gives

$$\frac{2}{\gamma M^2} \frac{dp}{p} + \frac{d(u^2)}{u^2} = -\frac{4f dx}{D} \quad (24)$$

From equation (22) we can get a relation that expresses the pressure derivative term, dp/p , in terms of Mach number and $d(u^2)/u^2$. Inserting this in (24) gives

$$\frac{2}{\gamma M^2} \left\{ - \left[\frac{1 + (\gamma - 1)M^2}{2} \right] \frac{d(u^2)}{u^2} \right\} + \frac{d(u^2)}{u^2} = -\frac{4f dx}{D} \quad (25)$$

collecting terms and rearranging gives

$$\frac{M^2 - 1}{\gamma M^2} \frac{d(u^2)}{u^2} = \frac{4f dx}{D} \quad (26)$$

if we now use equation (15) to get rid of the $d(u^2)/u^2$ -term we end up with an expression corresponding to equation (3.96)

$$\frac{4f dx}{D} = \frac{2}{\gamma M^2} (1 - M^2) \left[1 + \frac{(\gamma - 1)}{2} M^2 \right]^{-1} \frac{dM}{M} \quad (3.96)$$