Compressible Flow TME085

One-Dimensional Steady Flow

Flow with Added Heat

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Added Heat Relations

The aim is to derive relations for pressure ratio and temperature ratio as a function of Mach numbers. We will do that starting from the momentum equation.

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 \tag{1}$$

Assuming calorically perfect gas

$$\rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma p}{\rho} M^2 = \gamma p M^2$$

which inserted in Eqn. 1 gives

$$p_2 - p_1 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2$$

$$p_2(1 + \gamma M_2^2) = p_1(1 + \gamma M_1^2)$$

and thus

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{2}$$

From the equation of state $p = \rho R T$, we get

$$\frac{T_2}{T_1} = \frac{p_2}{\rho_2 R} \frac{\rho_1 R}{p_1} = \frac{p_2}{p_1} \frac{\rho_1}{\rho_2} \tag{3}$$

Using the continuity equation, we can get ρ_1/ρ_2

$$\rho_1 u_1 = \rho_2 u_2 \Rightarrow \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}$$

Inserted in Eqn. 3 gives

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{u_2}{u_1} \tag{4}$$

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} = \frac{M_2}{M_1}\frac{\sqrt{\gamma R T_2}}{\sqrt{\gamma R T_1}} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} \tag{5}$$

Eqn. 5 in Eqn. 4 gives

$$\sqrt{\frac{T_2}{T_1}} = \frac{p_2}{p_1}\frac{M_2}{M_1} \tag{6}$$

With p_2/p_1 from Eqn. 31, Eqn 6 becomes

$$\frac{T_2}{T_1} = \left(\frac{1+\gamma M_1^2}{1+\gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \tag{7}$$

Differential Relations

The equations presented in the previous section gives us the flow state after heat addition but since the heat addition, unlike the normal shock, is a continuous process, it is of interest to study the the heat addition from start to end. In order to do so we will now derive differential relations starting from the governing equations on differential form. We will start with converting the integral equation for conservation of mass for one-dimensional flows to differential form.

$$\rho_1 u_1 = \rho_2 u_2 = const \Rightarrow d(\rho u) = 0 \tag{8}$$

$$d(\rho u) = \rho du + u d\rho = 0$$

Divide by ρu gives

$$\frac{d\rho}{\rho} = \frac{du}{u} \tag{9}$$

The integral form of the conservation of momentum equation for one-dimensional flows is converted to differential form as follows.

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 = const \Rightarrow d(p + \rho u^2) = 0 \tag{10}$$

$$dp + \rho u du + u \underbrace{d(\rho u)}_{=0} = 0 \Rightarrow dp = -\rho u du \tag{11}$$

with $\rho = \dfrac{p}{RT}$ and $u^2 = M^2 a^2 = M^2 \gamma RT$ in Eqn. 11, we get

$$dp = -\frac{p}{RT} u^2 \frac{du}{u} = -\frac{p}{RT} M^2 \gamma RT \frac{du}{u} \Rightarrow$$

$$\frac{dp}{p} = -\gamma M^2 \frac{du}{u} \tag{12}$$

which gives the relative change in pressure, dp/p, as a function of the relative change in flow velocity, du/u. The next equation to derive is an equation that describes the relative change in temperature, dT/T, as a function of the relative change in flow velocity, du/u. The starting point is the equation of state (the gas law).

$$p = \rho RT \Rightarrow dp = R(\rho dT + T d\rho) \Rightarrow dT = \frac{1}{R\rho} dp - \frac{T}{\rho} d\rho \tag{13}$$

Divide by T

$$\frac{dT}{T} = \frac{1}{\rho RT} dp - \frac{1}{\rho} d\rho \tag{14}$$

substitute dp from Eqn. 12 and $d\rho$ from Eqn. 9 gives

$$\frac{dT}{T} = (1 - \gamma M^2) \frac{du}{u} \tag{15}$$

The entropy equation reads

$$ds = C_v \frac{dp}{p} - C_p \frac{d\rho}{\rho} \tag{16}$$

which after substituting dp from Eqn. 12 and $d\rho$ from Eqn. 9 becomes

$$ds = C_v\gamma(1-M^2)\frac{du}{u} \tag{17}$$

From the definition of total temperature T_o we get

$$T_o = T + \frac{u^2}{2C_p} \Rightarrow dT_o = dT + \frac{1}{C_p}udu = dT + \frac{\gamma-1}{\gamma RT}Tu^2\frac{du}{u} \Rightarrow$$

$$dT_o = dT + (\gamma-1)M^2T\frac{du}{u} \tag{18}$$

Inserting dT from Eqn 15 in Eqn 18 we get

$$dT_o = (1-\gamma M^2)T\frac{du}{u} + (\gamma-1)M^2T\frac{du}{u}$$

or

$$dT_o = (1-M^2)T\frac{du}{u} \tag{19}$$

Dividing Eqn. 19 by T_o and using

$$T_o = T\left(1+\frac{\gamma-1}{2}M^2\right)$$

we get

$$\frac{dT_o}{T_o} = \frac{1-M^2}{1+\dfrac{\gamma-1}{2}M^2}\frac{du}{u} \tag{20}$$

Finally, we will derive a differential relation that describes the change in Mach number.

$$M = \frac{u}{\sqrt{\gamma RT}} \Rightarrow dM = \frac{1}{\sqrt{\gamma R}}(T^{1/2}du + ud(T^{-1/2})) = \frac{du}{\sqrt{\gamma RT}} - \frac{u}{2\sqrt{\gamma R}}T^{-3/2}dT \Rightarrow$$

$$dM = \frac{1}{\sqrt{\gamma RT}}\frac{du}{u} - \frac{1}{2}\frac{u}{\sqrt{\gamma RT}}\frac{dT}{T} = M\frac{du}{u} - \frac{M}{2}\frac{dT}{T}$$

Inserting dT/T from Eqn. 15, we get

$$\frac{dM}{M} = \frac{1+\gamma M^2}{2}\frac{du}{u} \tag{21}$$

All the derived differential relations are expressed as functions of du/u but it would be more convenient to relate the changes in flow properties to the added heat or the change in total temperature, which can be related to the added heat through the energy equation.

$$dT_o = \frac{\delta q}{C_p}$$

From Eqn. 20, we get

$$\frac{du}{u} = \left(\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\right)\frac{dT_o}{T_o} \tag{22}$$

Now, we can substitute du/u in all the above relations using Eqn. 22, we get the following relations

$$\frac{d\rho}{\rho} = -\left(\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\right)\frac{dT_o}{T_o} \tag{23}$$

$$\frac{dp}{p} = \gamma M^2\left(\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\right)\frac{dT_o}{T_o} \tag{24}$$

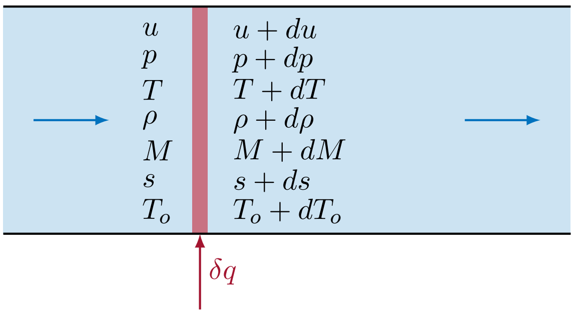
$$\frac{dT}{T} = (1-\gamma M^2)\left(\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\right)\frac{dT_o}{T_o} \tag{25}$$

$$\frac{dT}{T} = \left(\frac{1+\gamma M^2}{2}\right)\left(\frac{1+\frac{\gamma-1}{2}M^2}{1-M^2}\right)\frac{dT_o}{T_o} \tag{26}$$

$$\frac{dT}{T} = C_v\gamma\left(1+\frac{\gamma-1}{2}M^2\right)\frac{dT_o}{T_o} \tag{27}$$

Heat Addition Process

With the differential relations in place, we can now study the continuous change in flow quantities from the initial flow state to the flow state after the heat addition process by dividing the total amount of heat added to the flow, q, into small portions, δq, and calculate the change in flow properties for each of these heat additions, see Figure 1.



Figur 1: Change in flow quantities due to the addition of an infinitesimal amount of heat δq

Let's first examine the temperature change by rewriting Eqn. 25 as

$$dT = \frac{1-\gamma M^2}{1-M^2} dT_o \Leftrightarrow \frac{dT}{dT_o} = \frac{1-\gamma M^2}{1-M^2} \tag{28}$$

which is equivalent to

$$\frac{dh}{\delta q} = \frac{1-\gamma M^2}{1-M^2} \tag{29}$$

Form Eqn. 28 we can make the following observation

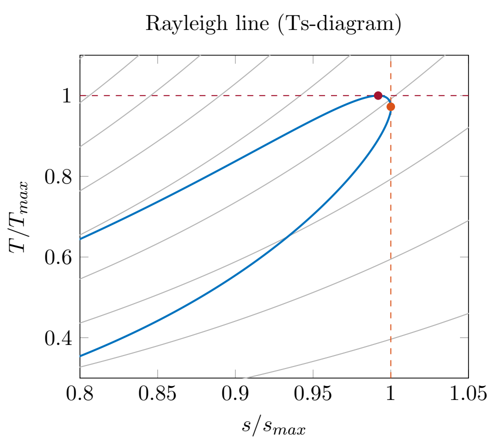
$$\frac{dT}{dT_o} = 0 \Rightarrow \gamma M^2 = 1 \Rightarrow M = \sqrt{1/\gamma}$$

which means that the maximum temperature will be reached when the Mach number is $\sqrt{1/\gamma}$. Since γ is a number grreter than one for all gases, this implies that the maximum temperature can only be reached if the flow is subsonic. For air, this the maximum temperature will be reached at $M = 0.845$.

If we evaluate Eqn. 28 for sonic flow ($M = 1$), we see that the derivative becomes infinite.

$$|M| \rightarrow 1.0 \Rightarrow \frac{dT}{dT_o} \rightarrow \pm\infty$$

Now, by specifying an initial subsonic flow state and dividing the heat addition corresponding to choked flow, q^*, into small portions δq, one can perform integration as indicated in Figure 1. The result is presented in the in Figure 2. The subsonic process corresponds to the upper line. As heat is added the Mach number is increased and at $M = \gamma^{-1/2}$ the maximum temperature is reached. Adding more heat will reduce the temperature and increase the Mach number until sonic conditions are reached ($M = 1.0$). As can be seen in Figure 2, the lean of the subsonic branch of the Rayleigh line is lower than the isobars (gray lines), which means the increasing heat will reduce pressure. The lower part of the blue line in Figure 2 is the supersonic branch of the Rayleigh line, which is obtained in the same way starting from a supersonic flow condition. A flow state resulting in the same sonic conditions as for the subsonic case is calculated and used as a starting state. The corresponding q^* is calculated and the same calculation of consecutive flow states in a step-wise manner is performed. As can be seen in Figure 2, the lean of the supersonic part of the Rayleigh curve is steeper than the isobars (gray lines), which means that pressure increases as heat is added to the flow. As we saw from Eqn. 29, dT/dT_o becomes infinite when the flow approaches the sonic the sonic state. After the sonic state is reached, further heat addition is impossible without changing the upstream flow conditions. This will be made clearer in the next section.



Figur 2: The Rayleigh line (the blue line) in a Ts-diagram. Gray lines are constant-pressure lines (isobars).

Rayleigh Line

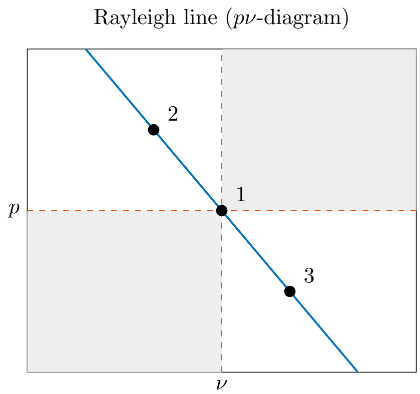
The continuity equation for steady-state, one-dimensional flow reads

$$\rho_1 u_1 = \rho_2 u_2 = C \tag{30}$$

where C is the massflow per square meter (massflow divided by area). Inserted in the momentum equation we get

$$p_1 + \frac{C^2}{\rho_1} = p_2 + \frac{C^2}{\rho_2} \Leftrightarrow p_1 + \nu_1 C^2 = p_2 + \nu_2 C^2 \Rightarrow \frac{p_2 - p_1}{\nu_2 - \nu_1} = -C^2 \tag{31}$$

Eqn. 31 tells us that any solution to the governing flow equations must lie along a line (a so-called Rayleigh line) in a $p\nu$-diagram. In Figure 3, 1 corresponds to the flow state before heat addition and states 2 and 3 corresponds to the flow state after heat is added. If the flow in state 1 is subsonic, adding heat will change the flow state following the Rayleigh line to the right, i.e. towards flow state 2. If the initial flow state instead is supersonic, heat addition will move the flow state towards state 3.



Figur 3: The Rayleigh line in a $p\nu$-diagram. The heat addition process will follow the Rayleigh line, i.e. all solutions to the equations will be on the line. The lean of the line depends on the massflow.

Now we know in which direction we will move along the Rayleigh curve when heat is added but in order to find the flow state after heat addition we need to add the energy equation to the problem. If we draw a curve corresponding to the energy equation including the heat addition in the same $p\nu$-diagram, the intersection of this curve and the Rayleigh line corresponds to the downstream flow state (the flow state that fulfils the continuity, momentum, and energy equations). To be able to do this we will rewrite the energy equation such that it can be represented by a line in the $p\nu$-diagram.

The energy equation for one-dimensional flow with heat addition reads

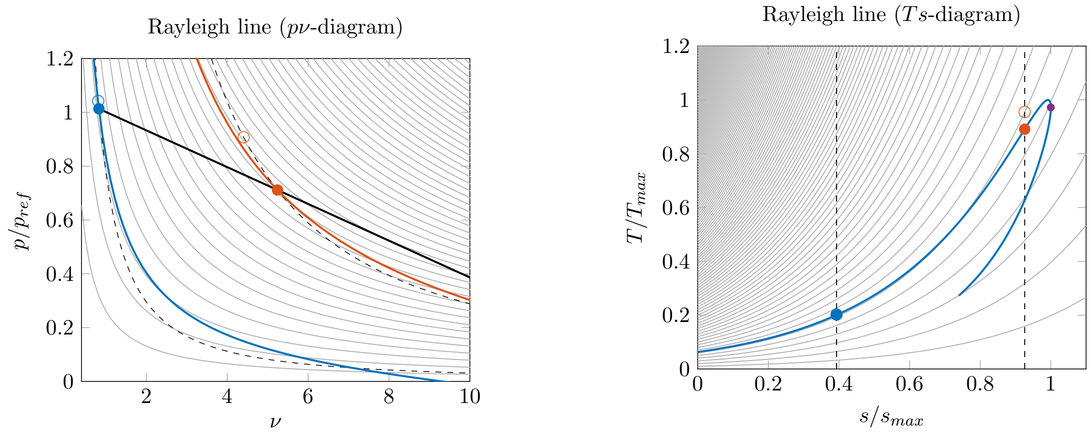
$$h_1 + \frac{1}{2}u_1^2 + q = h_2 + \frac{1}{2}u_2^2 \tag{32}$$

Inserting the constant C from above (the massflow per m^2) and and and $h = C_pT$, we get

$$\frac{\gamma R}{\gamma - 1}T_1 + \frac{1}{2}C^2\nu_1^2 + q = \frac{\gamma R}{\gamma - 1} + \frac{1}{2}C^2\nu_2^2 \tag{33}$$

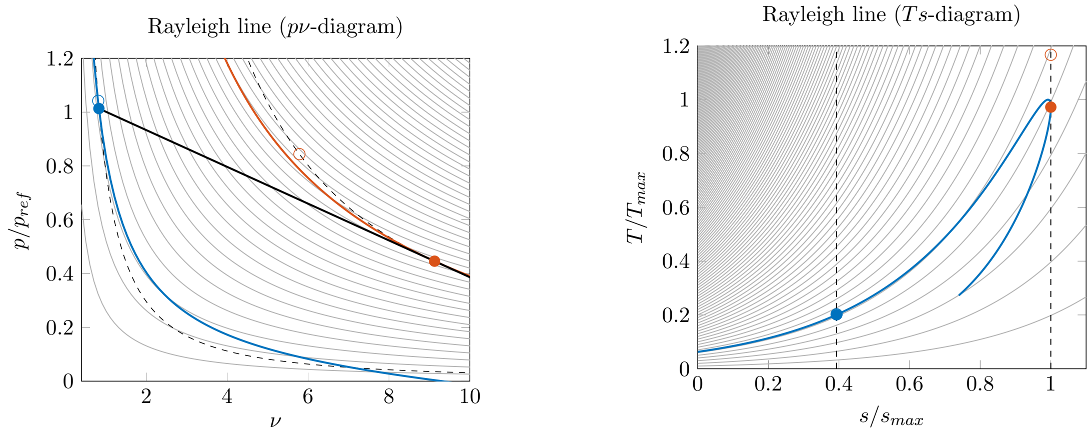
which may be rewritten as

$$\frac{p_2}{p_1} = \left(\frac{\nu_2}{\nu_1} - \frac{\gamma+1}{\gamma-1} - \frac{2q}{RT_1}\right)\left(1 - \frac{\gamma+1}{\gamma-1}\frac{\nu_2}{\nu_1}\right)^{-1} \tag{34}$$



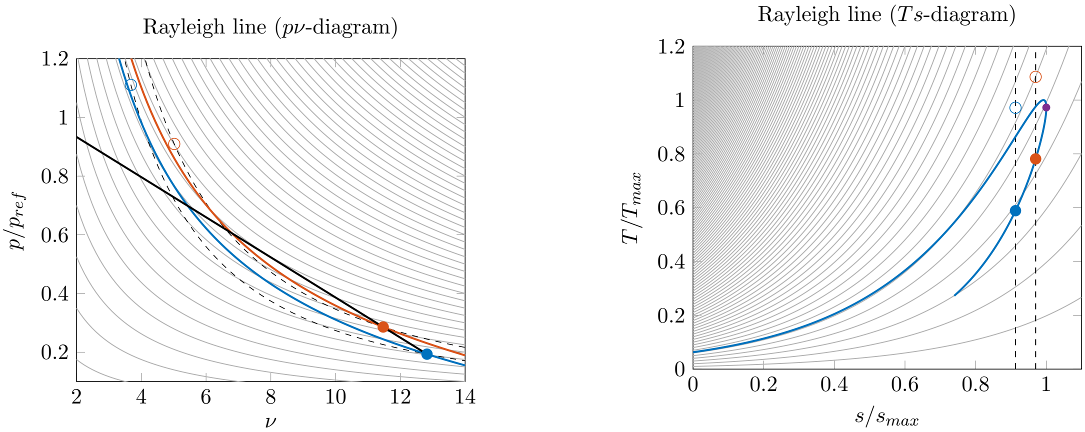
(a) $p\nu$-diagram **(b)** Ts-diagram

Figur 4: Subsonic heat addition. The filled blue circle denotes the start condition ($M_1 = 0.2$) and the filled orange circle denotes the flow state after heat addition ($M_2 = 0.6$). Blue and orange circles denote the corresponding total pressure (left) and total temperature (right), respectively. Dashed lines are isentropes (lines with constant entropy). Gray lines are isotherms (left) and isobars (right), respectively. In the left figure, the solid black line is the Rayleigh line, the solid blue line corresponds to the energy equation before heat addition and the solid orange line represents the energy equation after heat addition.



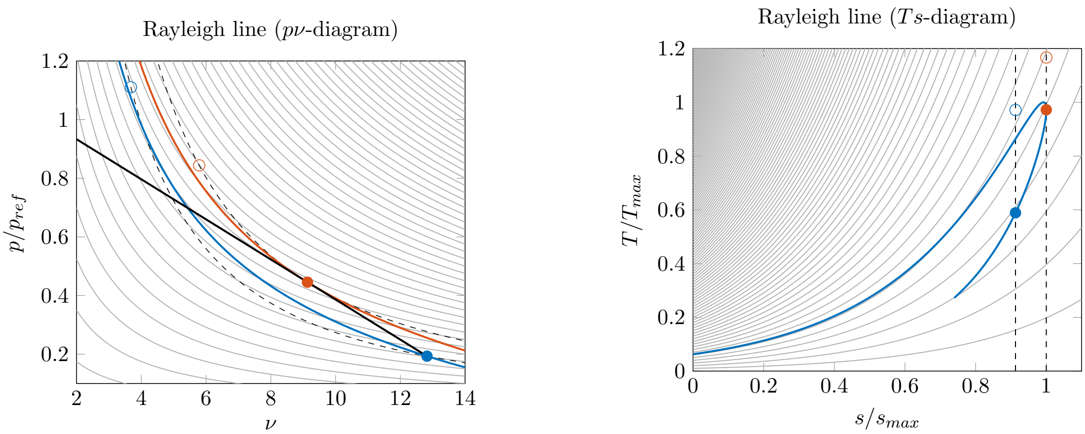
(a) $p\nu$-diagram **(b)** Ts-diagram

Figur 5: Subsonic heat addition. The filled blue circle denotes the start condition ($M_1 = 0.2$) and the filled orange circle denotes the flow state after heat addition ($M_2 = 1.0$). Blue and orange circles denote the corresponding total pressure (left) and total temperature (right), respectively. Dashed lines are isentropes (lines with constant entropy). Gray lines are isotherms (left) and isobars (right), respectively. In the left figure, the solid black line is the Rayleigh line, the solid blue line corresponds to the energy equation before heat addition and the solid orange line represents the energy equation after heat addition.



(a) $p\nu$-diagram **(b)** Ts-diagram

Figur 6: Supersonic heat addition. The filled blue circle denotes the start condition ($M_1 = 1.8$) and the filled orange circle denotes the flow state after heat addition ($M_2 = 1.4$). Blue and orange circles denote the corresponding total pressure (left) and total temperature (right), respectively. Dashed lines are isentropes (lines with constant entropy). Gray lines are isotherms (left) and isobars (right), respectively. In the left figure, the solid black line is the Rayleigh line, the solid blue line corresponds to the energy equation before heat addition and the solid orange line represents the energy equation after heat addition.



(a) $p\nu$-diagram **(b)** Ts-diagram

Figur 7: Supersonic heat addition. The filled blue circle denotes the start condition ($M_1 = 1.8$) and the filled orange circle denotes the flow state after heat addition ($M_2 = 1.0$). Blue and orange circles denote the corresponding total pressure (left) and total temperature (right), respectively. Dashed lines are isentropes (lines with constant entropy). Gray lines are isotherms (left) and isobars (right), respectively. In the left figure, the solid black line is the Rayleigh line, the solid blue line corresponds to the energy equation before heat addition and the solid orange line represents the energy equation after heat addition.

As you can see in the examples above (Figures 5 and 7), sonic conditions are reached when the Rayleigh line is tangent to the curve representing the energy equation in the $p\nu$-diagram. Adding more heat would move the energy equation line upwards and thus there can not be any solution after reaching this state unless the upstream conditions are changed such that the energy line intersects the Rayleigh line after further heat addition. Let's have a second look at the equations and see if it is possible to verify that the case where the Rayleigh line is a tangent to the energy-equation curve is in fact the sonic state.

Starting from Eqn. 33, it is easy to see that for any point along the energy equation curve the flow state may be expressed as a function of the initial flow state and the added heat q as

$$\frac{\gamma}{\gamma-1}p\nu+\frac{1}{2}C^2\nu^2=\frac{\gamma}{\gamma-1}p_1\nu_1+\frac{1}{2}C^2\nu_1^2+q=D \tag{35}$$

where D is a constant.

Now, let's different the Eqn.35 with respect to ν

$$\frac{\gamma}{\gamma-1}\left(\nu\frac{dp}{d\nu}+p\right)+C^2\nu=0\Rightarrow\frac{dp}{d\nu}=-\frac{\gamma}{\gamma-1}C^2-\frac{p}{\nu} \tag{36}$$

The Rayleigh line is a tangent to the energy equation curve when $dp/d\nu=-C^2$ and thus

$$\frac{C^2}{\gamma}=\frac{p}{\nu} \tag{37}$$

By definition $C=\rho u$ and $\nu=1/\rho$, which inserted in Eqn. 37 gives

$$u=\sqrt{\frac{\gamma p}{\rho}}=a \tag{38}$$